

THE ORDER STRUCTURE OF REGULAR OPERATORS BETWEEN BANACH LATTICES

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ABSTRACT. We present an up-to-date account of the state of knowledge of the order structure of the space of regular operators between two Banach lattices. As preliminaries, we present some notions of order theoretic completeness that arise in the statement of the results and results on the existence of Rademacher families in Banach lattices with an order continuous norm and on sequential weak* continuity of the lattice operations in their duals which are used in the proofs. We then look separately at the questions of when the space of regular operators in a vector lattice and when it has the Riesz Separation Property. We make no attempt here to attribute results to their original authors.

1. ORDER THEORETIC COMPLETENESS

I presume that everyone is familiar with the notions of Dedekind completeness and Dedekind σ -completeness. It turns out that when we try to characterize exactly when the space of regular operators between two Banach lattices is itself a lattice other variants of Dedekind completeness arise that we cannot avoid. In this section we introduce the appropriate definitions and give some variants of results that are known in the classical versions. The variants involve infinite cardinals other than $\aleph_0$ and even their properties turn out to be significant in what will follow.

Definition 1.1. The *cofinality* of an infinite cardinal $\mathfrak{a}$, $\text{cf}(\mathfrak{a})$, is the least cardinality of a cofinal subset of $\mathfrak{a}$. A cardinal $\mathfrak{a}$ is *singular* if $\text{cf}(\mathfrak{a}) < \mathfrak{a}$. A cardinal that is not singular is called *regular*.

Definition 1.2. An uncountable cardinal $\mathfrak{a}$ is *weakly inaccessible* if it is a regular limit cardinal. Their existence cannot be proved inside Zermelo-Fraenkel set theory together with the Axiom of Choice.

We commence our study of completeness conditions by making our definitions explicit.

Definition 1.3. Let $\mathfrak{a}$ be an infinite cardinal. A vector lattice X is *Dedekind $\mathfrak{a}$ -complete* (respectively *Dedekind $<\mathfrak{a}$ -complete*) if every nonempty subset of X , of cardinality at most (respectively less than) $\mathfrak{a}$ and which is bounded above, must have a supremum.

Of course a vector lattice is Dedekind complete if and only if it is Dedekind $\mathfrak{a}$ -complete for all cardinals $\mathfrak{a}$. Having made these definitions, we should show that there are some real examples. That turns out not to be elementary, so we need to defer it until we have proved some characterizations. In the mean time we obtain results that allow us to characterise these notions. As for the classical results the starting point is going to be what happens for $C(K)$, were we need to generalize the classical results of Nakano.

Definition 1.4. Let K be a topological space and $\mathfrak{a}$ an infinite cardinal.

- (i) A subset A of K is an $F_{\mathfrak{a}}$ set if there is a family $\mathfrak{F}$ of closed subsets of K such that $\text{card}(\mathfrak{F}) \leq \mathfrak{a}$ and $A = \bigcup \mathfrak{F}$.
- (ii) We call K $\mathfrak{a}$ -extremally disconnected if the closure of every open $F_{\mathfrak{a}}$ set is open. A $\mathfrak{a}$ -extremally disconnected compact Hausdorff space will be termed $\mathfrak{a}$ -Stonean.
- (iii) K is said to be $<\mathfrak{a}$ -extremally disconnected if K is $\mathfrak{b}$ -extremally disconnected for every $\mathfrak{b} < \mathfrak{a}$.

In the next result, $\mathbb{R}^*$ denotes the usual extended real number system.

Theorem 1.5. *If K is a normal topological space and $\mathfrak{a}$ an infinite cardinal then the following assertions are equivalent:*

- (i) K is $\mathfrak{a}$ -extremally disconnected.
- (ii) $C(K, \mathbb{R}^*)$ is an $\mathfrak{a}$ -complete lattice.
- (iii) $C(K)$ is a Dedekind $\mathfrak{a}$ -complete vector lattice.
- (iv) $C_b(K)$ is a Dedekind $\mathfrak{a}$ -complete vector lattice.

Proof. Suppose that K is card-extremally disconnected and that A is any non-empty subset of $C(K, \mathbb{R}^*)$. Let p be the pointwise supremum of A ,

$$p(k) = \sup\{f(k); f \in A\},$$

which need not, of course, be continuous. We write $\mathcal{N}_k$ for the filter of neighbourhoods of k , and define

$$a(k) = \inf_{N \in \mathcal{N}_k} \sup_{j \in N} p(j).$$

As $k \in N$ for each $N \in \mathcal{N}_k$, we see that $p(k) \leq \sup_{j \in N} p(j)$ for every such N so that $a(k) \geq p(k)$.

We now prove that any continuous upper bound for p is necessarily an upper bound for a . If not, there is a continuous $\mathbb{R}^*$ -valued function f_0 on K and $k_0 \in K$ such that $p \leq f_0$ but $f_0(k_0) < a(k_0)$. Choose $\epsilon > 0$ such that $f_0(k_0) + \epsilon < a(k_0)$. For every $N \in \mathcal{N}_{k_0}$ we certainly have $f_0(k_0) + \epsilon < \sup_{j \in N} p(j)$. In particular, we can choose $j_N \in N$ such that $f_0(k_0) + \epsilon < p(j_N) \leq f_0(j_N)$. The continuity of f_0 tells us that the set $\{k \in K : f_0(k) < f_0(k_0) + \epsilon\}$ is open, so is a neighbourhood of k_0 , N_0 say. Hence $f_0(j_{N_0}) < f_0(k_0) + \epsilon$, which contradicts the earlier result that $f_0(k_0) + \epsilon < f_0(j_N)$ for each $N \in \mathcal{N}_{k_0}$.

It remains to show that a itself is continuous. This will be accomplished if we can show that, for any $r \in \mathbb{R}$, each of the sets $L_r = \{k \in K : a(k) < r\}$ and $U_r = \{k \in K : a(k) > r\}$ is open. We begin with L_r . If $k \in K$ and $a(k) < r$ then, in view of the definition of a , there is $N \in \mathcal{N}_k$ such that $\sup_{j \in N} p(j) < r$. Choosing an open set G such that $k \in G \subset N$, it is clear that $\sup_{j \in G} p(j) < r$. As $G \in \mathcal{N}_j$, this implies that $a(j) < r$ for every $j \in G$. I.e. $j \in L_r$ and therefore L_r is open.

Next we consider U_r . For every natural number $n > 0$ let $D_n = \{k \in K : p(k) > r + \frac{1}{n}\}$ and let $W_n = K \setminus \overline{L_{r+\frac{1}{n}}}$. Clearly $D_n \subset W_n$ and hence $\overline{D_n} \subset W_n$ for all $n > 0$. If $k \in W_n$ then for every neighbourhood N of k we have $\sup_{j \in N} p(j) \geq r + \frac{1}{n}$ and there is $k_N \in N$ such that $k_N \in D_{n+1}$, which shows that $W_n \subset \overline{D_{n+1}}$, if $n > 0$. We can write D_n in two different ways:

$$\begin{aligned} D_n &= \bigcup_{f \in A} \{k \in K : f(k) > r + \frac{1}{n}\} \\ &= \bigcup_{f \in A} \bigcup_{m=1}^{\infty} \{k \in K : f(k) \geq r + \frac{1}{n} + \frac{1}{m}\}. \end{aligned}$$

As each f is continuous, the first expression shows that D_n is a union of open sets, so is open. Recalling that $\text{card}(A) \leq \mathfrak{a}$, the second expression shows that D_n is an $F_{\mathfrak{a}}$ set. As K is $\mathfrak{a}$ -extremally disconnected, $\overline{D_n}$ must be open. Now we see that

$$U_r = \bigcup_{n=1}^{\infty} W_n \subset \bigcup_{n=1}^{\infty} \overline{D_{n+1}} \subset \bigcup_{n=1}^{\infty} Q_{n+1} = U_r,$$

so that U_r is open.

We now know that a is continuous, so it is the supremum in $C(K, \mathbb{R})$ of A . A similar argument establishes the existence of the infimum of any nonempty family of cardinality at most $\mathfrak{a}$ so that $C(K, \mathbb{R}^*)$ is indeed card-complete. This proves that (i) implies (ii).

It is easy to see that either (ii) implies (iii) and clearly (iii) implies (iv). To complete the proof we need only prove that (iv) implies (i). So let us suppose that $C(K)$ is Dedekind $\mathfrak{a}$ -complete. Let U be an open $F_{\mathfrak{a}}$ subset of K , then there are closed sets $(U_{\alpha})_{\alpha \in \mathfrak{a}}$ such that $U = \bigcup_{\alpha \in \mathfrak{a}} U_{\alpha}$. As K is normal there are continuous functions $a_{\alpha} : K \rightarrow [0, 1]$ with $a_{\alpha}|_{U_{\alpha}} \equiv 1$ and $a_{\alpha}|_{K \setminus U} \equiv 0$. As the constantly one function is an upper bound for this family, assumption (iv) tells us that there is a supremum a in $C(K)$ of the family $(a_{\alpha})_{\alpha \in \mathfrak{a}}$. Clearly $a(k) \geq 1$ for all $k \in U$ and hence for all $k \in \overline{U}$. On the other hand if $k_0 \in K \setminus \overline{U}$ then there is a continuous function $b : K \rightarrow [0, 1]$ with $b|_{\overline{U}} \equiv 1$ and $b(k_0) = 0$. Then b is an upper bound for $(a_{\alpha})_{\alpha \in \mathfrak{a}}$ and hence

$b \geq a \geq 0$. Thus $0 = b(k_0) \geq a(k_0) \geq 0$. Thus a is zero on $K \setminus \overline{U}$ and is at least 1 on $\overline{U}$, so that $\overline{U}$ is open as required. $\square$

Corollary 1.6. *If K is a normal topological space and $\mathfrak{a}$ an infinite cardinal then the following assertions are equivalent:*

- (i) K is $<\mathfrak{a}$ -extremally disconnected.
- (ii) $C(K, \mathbb{R}^*)$ is an $<\mathfrak{a}$ -complete lattice.
- (iii) $C(K)$ is a Dedekind $<\mathfrak{a}$ -complete vector lattice.
- (iv) $C_b(K)$ is a Dedekind $<\mathfrak{a}$ -complete vector lattice.

The case that $\mathfrak{a} = \aleph_0$ of Theorem 1.9, which we will prove shortly, is due to Veksler and Gejler, [11]. The proof is slightly more technical in general. In particular we need:

Lemma 1.7. *Let K be a normal topological space which is $<\mathfrak{a}$ -extremally disconnected for some uncountable cardinal $\mathfrak{a}$. If $\mathfrak{A}$ is a family of open and closed subsets of K with $\text{card}(\mathfrak{A}) \leq \mathfrak{a}$ then there is a disjoint family $\mathfrak{B}$ of open and closed subsets of K with $\text{card}(\mathfrak{B}) \leq \mathfrak{a}$ and such that*

- (i) *Each member of $\mathfrak{B}$ is a subset of some member of $\mathfrak{A}$.*
- (ii) *$\bigcup \mathfrak{B}$ is dense in $\bigcup \mathfrak{A}$.*

Proof. Label the elements of $\mathfrak{A}$ as $(V_\alpha)_{\alpha \in \mathfrak{a}}$. For each $\beta \in \mathfrak{a}$ the set $\bigcup_{\gamma < \beta} V_\gamma$ is an open $F_\mathfrak{b}$ set, where $\mathfrak{b} = \text{card}(\beta) < \mathfrak{a}$, so has an open closure. Define

$$W_\beta = V_\beta \setminus \overline{\bigcup_{\gamma < \beta} V_\gamma}$$

for each $\beta \in \mathfrak{a}$. Each W_β is an open and closed subset of V_β . Define $\mathfrak{B} = (W_\beta)_{\beta \in \mathfrak{a}}$, which is a pairwise disjoint family of cardinality $\mathfrak{a}$ or less.

It remains only to show that $\bigcup \mathfrak{B}$ is dense in $\bigcup \mathfrak{A}$. If $k \in \bigcup \mathfrak{A}$ let α_0 be the smallest member of $\mathfrak{a}$ with $k \in V_{\alpha_0}$. If $k \notin W_{\alpha_0}$ then $k \in \overline{\bigcup_{\alpha < \alpha_0} V_\alpha}$. Let G be any open subset of K containing k and let α_1 be the smallest member of $\mathfrak{a}$ such that $G \cap V_{\alpha_1} \neq \emptyset$ (noting that $\alpha_1 < \alpha_0$.) We have $G \cap \bigcup_{\alpha < \alpha_1} V_\alpha = \emptyset$ and hence (as G is open) $G \cap \overline{\bigcup_{\alpha < \alpha_1} V_\alpha} = \emptyset$. It follows that $G \cap V_{\alpha_1} \subset G \cap W_{\alpha_1}$, proving that $G \cap W_{\alpha_1} \neq \emptyset$. It follows that every neighbourhood of k meets $\bigcup \mathfrak{B}$ as required. $\square$

Theorem 1.8. *If $\mathfrak{a}$ is an uncountable cardinal and K is a $\aleph_0$ -extremally disconnected normal topological space then the following are equivalent:*

- (i) K is $\mathfrak{a}$ -extremally disconnected
- (ii) *For every pairwise disjoint family $\mathfrak{A}$ of open and closed subsets of K , with $\text{card}(\mathfrak{A}) \leq \mathfrak{a}$, the closure of $\bigcup \mathfrak{A}$ is open.*

Proof. Clearly (i) implies (ii). If the converse implication were not true for all infinite cardinals $\mathfrak{a}$, let $\mathfrak{a}$ be the smallest cardinal for which it fails. Trivially $\mathfrak{b} > \aleph_0$. Let L be an $\aleph_0$ -extremally disconnected

normal topological space satisfying (ii). The minimality of $\mathfrak{a}$ ensures that L is $\mathfrak{b}$ -extremally disconnected for every $\mathfrak{b} < \mathfrak{a}$. If U is an open $F_{\mathfrak{a}}$ subset of L . We can write $U = \bigcup_{\alpha \in \mathfrak{a}} U_{\alpha}$ where each U_{α} is closed. By normality there is an open F_{σ} set V_{α} such that $U_{\alpha} \subset V_{\alpha} \subset U$. As L is $\aleph_0$ -extremally disconnected, $W_{\alpha} = \overline{V_{\alpha}}$ is open. Let $\mathfrak{A} = (W_{\alpha})_{\alpha \in \mathfrak{a}}$ and apply Lemma 1.7 to find a pairwise disjoint family of clopen sets $\mathfrak{B}$ of cardinality $\mathfrak{a}$, each contained in one of the sets W_{α} and with $\bigcup \mathfrak{B}$ dense in $\bigcup \mathfrak{A}$ and hence in $\overline{U}$. By (ii) $\bigcup \mathfrak{B}$ is open so that $\overline{U}$ is open and L is $\mathfrak{a}$ -extremally disconnected. $\square$

Theorem 1.9. *If X is a uniformly complete vector lattice and $\mathfrak{a}$ is an infinite cardinal then the following are equivalent:*

- (i) *X is Dedekind $\mathfrak{a}$ -complete.*
- (ii) *If $A \subset X_+$ is laterally increasing, bounded above and with $\text{card}(A) \leq \mathfrak{a}$ then A has a supremum in X .*
- (iii) *If $A \subset X_+$ is a pairwise disjoint order bounded set with $\text{card}(A) \leq \mathfrak{a}$ then A has a supremum in X .*

Proof. Clearly (i) implies (ii). If (ii) holds then we need only note that in (iii) the collection of all finite suprema from A is laterally increasing, has the same cardinality as A and has the same upper bounds as A to assert that (iii) holds.

Now suppose that (iii) holds. In order to prove that X is Dedekind $\mathfrak{a}$ -complete it suffices to prove that every principal ideal in X is Dedekind $\mathfrak{a}$ -complete, so we may restrict our attention to the case that $X = C(K)$ for some compact Hausdorff space K . The result is known when $\mathfrak{a} = \aleph_0$, [7], Corollary 2.1.6, so K is $\aleph_0$ -Stonean. By Theorem 1.8 in order to show that K is $\mathfrak{a}$ -Stonean it suffices to prove that if $\mathfrak{A}$ is a family of open and closed sets with $\text{card}(\mathfrak{A}) \leq \mathfrak{a}$ then $\bigcup \mathfrak{A}$ is open. The family of functions $A = \{\chi_V : V \in \mathfrak{A}\}$ is pairwise disjoint, bounded above by the constantly one function and has $\text{card}(A) \leq \mathfrak{a}$ so the family has a supremum a . This supremum is clearly at least one on $\bigcup \mathfrak{A}$ and hence on $\overline{\bigcup \mathfrak{A}}$ and given any $k_0 \in K \setminus \overline{\bigcup \mathfrak{A}}$ we can find a continuous function $b : K \rightarrow [0, 1]$ which is 1 on $\overline{\bigcup \mathfrak{A}}$ and with $b(k_0) = 0$. This will be an upper bound for A showing that $a(k_0) = 0$. This shows that $\overline{\bigcup \mathfrak{A}}$ is indeed open. $\square$

By taking all cardinals $\mathfrak{a}$ we obtain as a corollary the other of Veksler and Gejler's results from [11].

Corollary 1.10. *If X is a uniformly complete vector lattice then the following are equivalent:*

- (i) *X is Dedekind complete.*
- (ii) *If $A \subset X_+$ is laterally increasing and bounded above then A has a supremum in X .*
- (iii) *If $A \subset X_+$ is a pairwise disjoint order bounded set then A has a supremum in X .*

We can finally give the promised non-trivial examples of Dedekind $\mathfrak{a}$ -complete and Dedekind $<\mathfrak{a}$ -complete vector lattices.

Definition 1.11. Let $\mathfrak{a}$ be an infinite cardinal and Γ be a set of cardinality at least $\mathfrak{a}$. We denote by $c(\Gamma; \mathfrak{a})$ the space of all bounded functions $f : \Gamma \rightarrow \mathbb{R}$ such that there is a real number ℓ (denoted by $\lim_{\mathfrak{a}} f$) with the property that for all $\epsilon > 0$ the set $\{\gamma \in \Gamma : |f(\gamma) - \ell| > \epsilon\}$ has cardinality less than $\mathfrak{a}$.

It is routine to verify that $c(\Gamma; \mathfrak{a})$ is a Banach space under the supremum norm and is a vector lattice under the pointwise partial order (of course, it is actually a Banach lattice.)

Theorem 1.12. Let $\mathfrak{a}, \mathfrak{b}$ be infinite cardinals and Γ be a set of cardinality at least $\mathfrak{a}$. Then $c(\Gamma, \mathfrak{a})$ is Dedekind $\mathfrak{b}$ -complete if and only if $\mathfrak{b} < \text{cf}(\mathfrak{a})$.

Proof. Suppose that $\mathfrak{a}$ is strictly less than the cofinality of $\mathfrak{b}$. By 1.9 in order to show that $c(\Gamma; \mathfrak{a})$ is Dedekind $\mathfrak{b}$ -complete it suffices to prove that every pairwise disjoint order bounded family $A = (a_\beta)_{\beta \in \mathfrak{b}}$ in $c(\Gamma; \mathfrak{a})_+$ has a supremum. At most one of the functions a_β has $\lim_{\mathfrak{a}} a_\beta \neq 0$. We consider the case that $\lim_{\mathfrak{a}} a_\beta = 0$ for all $\beta \in \mathfrak{b}$ (take the supremum of its supremum plus one other element afterwards.)

Let a be the pointwise supremum of A , which we show lies in $c(\Gamma; \mathfrak{a})$. If $\epsilon > 0$ then

$$\begin{aligned} \{\gamma \in \Gamma : a(\gamma) > \epsilon\} &= \{\gamma \in \Gamma : \sup_{\beta \in \mathfrak{b}} a_\beta(\gamma) > \epsilon\} \\ &= \{\gamma \in \Gamma : a_\beta > \epsilon \text{ for some } \beta \in \mathfrak{b}\} \\ &= \bigcup_{\beta \in \mathfrak{b}} \{\gamma \in \Gamma : a_\beta(\gamma) > \epsilon\}. \end{aligned}$$

Each of the sets $\{\gamma \in \Gamma : a_\beta(\gamma) > \epsilon\}$ has cardinality less than $\mathfrak{a}$ since $\lim_{\mathfrak{a}} a_\beta = 0$. As $\mathfrak{b}$ is less than the cofinality of $\mathfrak{a}$ the union of $\mathfrak{b}$ sets of cardinality less than $\mathfrak{a}$ also has cardinality less than $\mathfrak{a}$. I.e. $\lim_{\mathfrak{a}} a = 0$ so that $a \in c(\Gamma; \mathfrak{a})$ as required.

Now suppose that $\mathfrak{b}$ is equal to the cofinality of $\mathfrak{a}$. Let Δ be a subset of Γ with Δ having cardinality $\mathfrak{a}$ and $\Gamma \setminus \Delta$ having cardinality at least $\mathfrak{a}$. We may partition Δ into nonempty subsets $(\Delta_\beta)_{\beta \in \mathfrak{b}}$ with each Δ_β having cardinality less than $\mathfrak{a}$. Clearly the characteristic function of each Δ_β lies in $c(\Gamma; \mathfrak{a})$. The family of all such characteristic functions is pairwise disjoint and bounded above by the constantly one function. The pointwise supremum is the characteristic function of Δ , χ_Δ , so does not lie in $c(\Gamma; \mathfrak{a})$. If a were the supremum of this family, it has to be at least as large as this pointwise supremum and cannot be equal to it, so for some $\gamma_0 \in \Gamma$ and $\epsilon > 0$ we have $a(\gamma_0) > \chi_\Delta(\gamma_0) + \epsilon$. If we were to modify a to a' by requiring that $a'(\gamma_0) = \chi_\Delta(\gamma_0)$ then we

still have $a' \in c(\Gamma; \mathfrak{a})$, a' still dominates the original disjoint family and is less than a , contradicting a being the supremum of the family. I.e. $c(\Gamma; \mathfrak{a})$ is not Dedekind $\mathfrak{b}$ -complete. $\square$

In particular, note that:

Example 1.13. If $\mathfrak{a}$ is a regular cardinal, then $c(\mathfrak{a}, \mathfrak{a})$ is Dedekind $<\mathfrak{a}$ -complete but not Dedekind $\mathfrak{a}$ -complete.

For singular cardinals, on the other hand, the two notions coincide. This needs one preliminary result. It is elementary to verify that a Banach lattice is Dedekind σ -complete if and only if every increasing sequence which is bounded above has a supremum. We can characterize Dedekind $\mathfrak{a}$ -completeness in a similar fashion.

Theorem 1.14. *For an infinite cardinal number $\mathfrak{a}$ the following conditions on a vector lattice X are equivalent:*

- (i) X is Dedekind $\mathfrak{a}$ -complete.
- (ii) *If η is an ordinal of cardinality at most $\mathfrak{a}$, $f : \eta \rightarrow X$ is an increasing function and $f(\eta)$ has an upper bound, then $f(\eta)$ has a supremum.*
- (iii) *If η is an initial ordinal of cardinality at most $\mathfrak{a}$, $f : \eta \rightarrow X$ is an increasing function and $f(\eta)$ has an upper bound, then $f(\eta)$ has a supremum.*

Proof. It is clear that (i) implies (ii), which in turn implies (iii). In order to establish that (iii) implies (i), we may suppose that there is some subset of X with an upper bound but no supremum (else (i) is certainly true). Let $\mathfrak{b}$ be the smallest cardinal of a set $B \subset X$ which is bounded above but has no supremum. Let η be the first ordinal of cardinality equal to $\mathfrak{b}$. Index B by the elements of η , so that $B = \{b_i : i < \eta\}$. For each $i < \eta$, $\text{card}(i) < \mathfrak{b}$, so that $f(i) = \sup\{b_j : j < i\}$ exists in X , by the definition of $\mathfrak{b}$. Now $f : \eta \rightarrow X$ is an increasing function which is bounded above. To establish (i) we need to prove that $\mathfrak{b} > \mathfrak{a}$. If not, then $\mathfrak{b} \leq \mathfrak{a}$ so that (iii) guarantees that $\sup f(\eta)$ exists. Clearly $\sup f(\eta) \geq f(i+1) \geq b_i$ for each $i \in \eta$, so that $\sup f(\eta)$ is an upper bound for B . On the other hand any upper bound c for B will also be an upper bound for each set $\{b_j : j < i\}$, so $c \geq f(i)$ and hence $c \geq \sup f(\eta)$. Thus $\sup f(\eta)$ is the supremum of B , contrary to hypothesis. Thus $\mathfrak{b} > \mathfrak{a}$ and hence (i) holds. $\square$

Theorem 1.15. *If $\mathfrak{a}$ is a singular cardinal then every Dedekind $<\mathfrak{a}$ -complete vector lattice X is Dedekind $\mathfrak{a}$ -complete.*

Proof. By Theorem 1.14 in order to prove that X is Dedekind $\mathfrak{a}$ -complete it suffices to prove that any set $S \subset X$, which is order isomorphic to the ordinal $\mathfrak{a}$ and bounded above, has a supremum in X . As $\mathfrak{a}$ is singular there is a cofinal subset $T \subset S$ with cardinality strictly less than $\mathfrak{a}$. As

X is Dedekind $<\mathfrak{a}$ -complete, T must have a supremum in X , which is clearly also the supremum of S . $\square$

2. ORDER CONTINUOUS NORMS

Our aim in this section is to show the existence of sufficiently large families in a Banach lattice with an order continuous norm having a property analogous to that of the classical family of Rademacher functions. It will be convenient to formulate our results in a homogeneous setting, which we first define.

Definition 2.1. If $\mathfrak{a}$ is an infinite cardinal then we say that a topological space is $\mathfrak{a}$ -separable if it contains a dense subset of cardinality at most $\mathfrak{a}$. The *density character* of a subset S of a topological space is the smallest cardinal $\mathfrak{a}$ such that S is $\mathfrak{a}$ -separable. So S is separable if its density character is $\aleph_0$, the first infinite cardinal. We denote the density character of S by $\mathfrak{d}(S)$.

Definition 2.2. Let X be any Banach lattice and S a non-empty order convex subset of X which is closed under $\wedge$ and $\vee$. If $\mathfrak{a}$ is an infinite cardinal we say that S is $\mathfrak{a}$ -homogeneous if

- (i) S is not a singleton set.
- (ii) S is disjoint from the atoms in X .
- (iii) If $x, y \in S$ with $x \leq y$ and $x \neq y$ then $\mathfrak{d}([x, y]) = \mathfrak{a}$.

The basic idea is that in some sense all these order intervals are of comparable size. The assumption that S is disjoint from the atoms in X is because we would otherwise be describing $\ell_1 \oplus L_1([0, 1])$ as $\aleph_0$ -homogeneous, which we would clearly not want to do.

Recall that a measure algebra (Σ, μ) is naturally metrisable by defining $d(A, B) = \mu(A \setminus B) + \mu(B \setminus A)$. The measure algebra (Σ, μ) is said to be $\mathfrak{a}$ -homogeneous if every subset with non-zero measure has density character $\mathfrak{a}$. We will denote by γ the measure on the two point discrete space $\mathbf{2} = \{0, 1\}$ which satisfies $\gamma(\{0\}) = \gamma(\{1\}) = \frac{1}{2}$. If $\mathfrak{a}$ is any infinite cardinal then $\gamma^\mathfrak{a}$ denotes the product of $\mathfrak{a}$ many copies of γ on the Borel subsets of $\mathbf{2}^\mathfrak{a}$. The measure algebra consisting of $\gamma^\mathfrak{a}$ on the measurable subsets of $\mathbf{2}^\mathfrak{a}$ is, up to isomorphism, the only $\mathfrak{a}$ -homogeneous measure algebra. It is a routine exercise to show that a measure algebra (Σ, μ) is $\mathfrak{a}$ -homogeneous if and only if $L_1(\mu)$ is $\mathfrak{a}$ -homogeneous.

There are thus Banach lattices which are $\mathfrak{a}$ -homogeneous for any infinite cardinal $\mathfrak{a}$. If $\mathfrak{A}$ is any non-empty set of cardinals which is bounded above then the Banach lattice $\ell_1(L_1(\mathbf{2}^\mathfrak{a}, \gamma^\mathfrak{a}); \mathfrak{a} \in \mathfrak{A})$ contains bands which are $\mathfrak{a}$ -homogeneous precisely when $\mathfrak{a} \in \mathfrak{A}$.

Maharam's representation of measure algebras in terms of homogeneous measure algebras and of the latter in terms of algebras of the form $(\mathbf{2}^\mathfrak{a}, \gamma^\mathfrak{a})$ allows the following very concrete description of AL-spaces. This is a little used alternative to Kakutani's classical representation

of AL-spaces. See, for example, [10] §26 for details. In the statement of the following result, $\mathfrak{a}Y$ denotes the ℓ_1 -direct sum of $\mathfrak{a}$ many copies of Y .

Theorem 2.3. *Let X be an AL-space. There exists a unique well-ordered family $(\mathfrak{a}_\sigma)_{-1 \leq \sigma < \tau}$ such that:*

- (i) *for each $\sigma \geq 0$, each $\mathfrak{a}_\sigma$ is equal to 0, or to 1, or is uncountable.*
- (ii) *$\{\sigma : \mathfrak{a}_\sigma \neq 0\}$ is cofinal in τ , and*
- (iii) *Y is isometrically order isomorphic to $\ell_1^{\mathfrak{a}-1} \oplus \ell_1(\mathfrak{a}_\sigma L_1(2^{\aleph_\sigma}, \gamma^{\aleph_\sigma}); 0 \leq \sigma < \tau)$.*

Corollary 2.4. *Let X be a Banach lattice with an order continuous norm. There is a unique ordinal τ , a cofinal subset Σ of τ and a pairwise disjoint collection $(X_\sigma)_{\sigma \in \Sigma}$ of bands in X such that X_σ is $\aleph_\sigma$ -homogeneous and $X = \text{at}(X) \oplus \sum_{\sigma \in \Sigma} X_\sigma$, where $\text{at}(X)$ is the band generated by the atoms in X .*

Proof. If X had a weak order unit then we could find a probability measure μ such that $L_\infty(\mu) \subseteq X \subseteq L_1(\mu)$, where the embedding of $L_\infty(\mu)$ is onto a dense ideal in X and that of X is onto a dense ideal in $L_1(\mu)$, [7], Theorem 2.7.8. In general, by producing a maximal disjoint family in X_+ we obtain an embedding of X onto a dense ideal in an ℓ_1 -sum of spaces $L_1(\mu)$. I.e. X may be identified with a dense ideal in an AL-space, which may be described as in the preceding theorem. Take the same τ , $\Sigma = \{\sigma : \sigma \geq 0 \text{ and } \mathfrak{a}_\sigma > 0\}$ and $X_\sigma = X \cap \mathfrak{a}_\sigma L_1(2^{\aleph_\sigma}, \gamma^{\aleph_\sigma})$. The space $\mathfrak{a}_\sigma L_1(2^{\aleph_\sigma}, \gamma^{\aleph_\sigma})$ is $\aleph_\sigma$ -homogeneous and the theorem of Amemiya, [7] Theorem 2.4.8, tells us that the norms in the AL-space and in X generate the same topology on order intervals in X so that X_σ is also $\aleph_\sigma$ -homogeneous. $\square$

Note that the bands X_σ have a weak order unit if and only if the corresponding $\mathfrak{a}_\sigma$ is 1, rather than being uncountable.

The $\mathfrak{a}$ -homogeneous Banach lattices with order continuous norms obviously warrant more detailed attention. Those with a weak order unit e are especially nice. There is $\phi \in X_+^*$ such that $\phi(e) = 1$ and with $\phi(x) > 0$ if $0 \neq x \in X_+$. We call such a ϕ an *e-integral*. The proof of the preceding corollary, together with the more detailed statement of Theorem 2.7.8 of [7] and Maharam's characterization of homogeneous measure algebras, shows that:

Proposition 2.5. *Let $\mathfrak{a}$ be an infinite cardinal and let X be an $\mathfrak{a}$ -homogeneous Banach lattice with an order continuous norm, a weak order unit e and an e -integral ϕ . There are injective lattice homomorphisms $\Pi : X \rightarrow L_1(2^\mathfrak{a}, \gamma^\mathfrak{a})$ and $\Theta : X^* \rightarrow L_1(2^\mathfrak{a}, \gamma^\mathfrak{a})$, such that $\Pi(e) = \mathbf{1}$ and $\Theta(\phi) = \mathbf{1}$ satisfying*

- (i) *$\Pi(X)$ is a dense ideal in $L_1(2^\mathfrak{a}, \gamma^\mathfrak{a})$.*
- (ii) *$L_\infty(2^\mathfrak{a}, \gamma^\mathfrak{a})$ is a dense ideal in $\Pi(X)$.*

- (iii) $\Theta(X^*)$ is a dense ideal in $L_1(\mathbf{2}^{\mathfrak{a}}, \gamma^{\mathfrak{a}})$.
- (iv) $L_{\infty}(\mathbf{2}^{\mathfrak{a}}, \gamma^{\mathfrak{a}})$ is an order dense ideal in $\Theta(X^*)$. Furthermore, if $x \in X$ and $f \in X^*$ then $f(x) = \int \Theta(f)\Pi(x)d\gamma^{\mathfrak{a}}$.

It is possible to create an abstract theory of systems that imitate the behaviour of the sequence of Rademacher functions on $[0, 1]$, but that is not necessary for our purposes. We only need the existence of systems with particularly nice behaviour rather than to prove that all such systems are nice. As that will shorten our exposition to a major extent, we will limit ourselves to that task.

Definition 2.6. Let X be an Banach lattice with an order continuous norm and with a weak order unit e and an e -integral ϕ . A non-empty family of pairs $\{(e_{\alpha}, \phi_{\alpha}) : \alpha \in \mathfrak{a}\} \subset X \times X^*$, where $\mathfrak{a}$ is an ordinal, is said to be a (ϕ, e) -Rademacher system if

- (i) For all $\alpha \in \mathfrak{a}$, $|\phi_{\alpha}| = \phi$ and $|e_{\alpha}| = e$.
- (ii) For all $\alpha, \beta \in \mathfrak{a}$, $\phi_{\alpha}(e) = \phi(e_{\alpha})$ and $\phi_{\alpha}(e_{\beta}) = \begin{cases} 1 & \text{if } \alpha = \beta \\ 0 & \text{if } \alpha \neq \beta. \end{cases}$
- (iii) If $\alpha \neq \beta$ then $|\phi_{\alpha} - \phi_{\beta}|(e) = \phi(|e_{\alpha} - e_{\beta}|) = \frac{1}{2}$.
- (iv) For all $x \in X$, all $f \in X^*$ and all $\epsilon > 0$ the sets $\{\alpha \in \mathfrak{a} : |\phi_{\alpha}(x)| \geq \epsilon\}$ and $\{\alpha \in \mathfrak{a} : |f(e_{\alpha})| \geq \epsilon\}$ are finite.
- (v) If $m \in \mathbb{N}$, $\sigma : \{1, 2, \dots, m\} \rightarrow \{+, -\}$ and we set $\psi_{\alpha} = \bigwedge_{j=1}^m \phi_{\alpha+j}^{\sigma(j)}$ then we have
 - (a) $\psi_{\alpha}(e) = 2^{-m}$ for all $\alpha \in \mathfrak{a}$, and
 - (b) for all $x \in X$ and all $\epsilon > 0$ the set $\{\alpha \in \mathfrak{a} : |\psi_{\alpha}(x) - 2^{-m}\phi(x)| \geq \epsilon\}$ is finite.

At present, we will limit ourselves to proving the existence of sufficiently large such systems and some very basic properties.

Theorem 2.7. If $\mathfrak{a}$ is an infinite cardinal and X is an $\mathfrak{a}$ -homogeneous Banach lattice with an order continuous norm, a weak order unit e and an e -integral ϕ then there is a (ϕ, e) -Rademacher system of cardinality $\mathfrak{a}$.

Proof. Represent X and X^* as in Proposition 2.5. For each $\alpha \in \mathfrak{a}$, let r_{α} denote the function on $\mathbf{2}^{\mathfrak{a}}$ which is 1 if the α 'th component is 0 and is -1 if that component is 1. This certainly gives us a family of cardinality $\mathfrak{a}$ and with $|r_{\alpha}| = \mathbf{1}$ for each $\alpha \in \mathfrak{a}$. For each $\alpha \in \mathfrak{a}$ let $\phi_{\alpha} = \Theta^{-1}(r_{\alpha})$, so that $|\phi_{\alpha}| = \phi$, and $e_{\alpha} = \Pi^{-1}(r_{\alpha})$, so that $|e_{\alpha}| = e$. The statements in (ii) and (iii) are clear from the representation.

It is clear that $(r_{\alpha})_{\alpha \in \mathfrak{a}}$ is an orthonormal system in $L_2(\mathbf{2}^{\mathfrak{a}}, \gamma^{\mathfrak{a}})$, so that for all $y \in L_2(\mathbf{2}^{\mathfrak{a}}, \gamma^{\mathfrak{a}})$ we have

$$\sum_{\alpha \in \mathfrak{a}} \left| \int r_{\alpha} y d\gamma^{\mathfrak{a}} \right|^2 < \infty.$$

In particular, for each $\epsilon > 0$, the set $F_y = \{\alpha \in \mathfrak{a} : |\int r_\alpha y d\gamma^\mathfrak{a}| \geq \epsilon/2\}$ is finite. If we take any $x \in X$ we can find $y \in L_\infty(\mathbf{2}^\mathfrak{a}, \gamma^\mathfrak{a}) \subset L_2(\mathbf{2}^\mathfrak{a}, \gamma^\mathfrak{a})$ with $\|\mathcal{P}(x) - y\|_1 < \epsilon/2$. For all α outside the finite set F_α we have

$$\begin{aligned} |\phi_\alpha(x)| & \left| \int r_\alpha \Pi(x) d\gamma^\mathfrak{a} \right| \\ & \leq \left| \int r_\alpha y d\gamma^\mathfrak{a} \right| + \left| \int r_\alpha (\Pi(x) - y) d\gamma^\mathfrak{a} \right| \\ & < \epsilon/2 + \int |r_\alpha| |\Pi(x) - y| d\gamma^\mathfrak{a} \\ & < \epsilon/2 + \|\Pi(x) - y\|_1 < \epsilon \end{aligned}$$

which completes half the proof of (iv). The proof of the other half is similar.

It is clear that $\bigwedge_{j=1}^m r_{\alpha+j}^{s(j)} = \prod_{j=1}^m t_{\alpha+j}^{s(j)}$ is the characteristic function of a set of $\gamma^\mathfrak{a}$ -measure 2^{-m} . It follows that

$$\begin{aligned} \psi_\alpha(e) &= \bigwedge_{j=1}^m \phi^{s(j)} \alpha + j(e) \\ &= \bigwedge_{j=1}^m \Theta^{-1}(r_{\alpha+j}^{s(j)} (\Pi^{-1}(\mathbf{1}))) \\ &= \Theta^{-1} \left(\bigwedge_{j=1}^m r_{\alpha+j}^{s(j)} \right) (\Pi^{-1}(\mathbf{1})) \\ &= \int \bigwedge_{j=1}^m r_{\alpha+j}^{s(j)} \times \mathbf{1} d\gamma^\mathfrak{a} = 2^{-m}, \end{aligned}$$

which establishes (v) (a).

If we set $t_\alpha = r_{\alpha+q_1} r_{\alpha+q_2} \dots r_{\alpha+q_k}$ (where $q_1 < q_2 < \dots < q_k$) and $\alpha < \beta$ then t_α is orthogonal to t_β in $L_2(\mathbf{2}^\mathfrak{a}, \gamma^\mathfrak{a})$, as the product $t_\alpha t_\beta$ is a product of powers of r_δ , not all of which are $\mathbf{1}$ as $r_{\alpha+q_1}$ only occurs once. It follows that (t_α) is an orthonormal sequence in $L_2(\mathbf{2}^\mathfrak{a}, \gamma^\mathfrak{a})$. Again it follows that for any $\epsilon > 0$ and $y \in L_2(\mathbf{2}^\mathfrak{a}, \gamma^\mathfrak{a})$ the set $\{\alpha \in \mathfrak{a} : |\int t_\alpha y d\gamma^\mathfrak{a}| \geq \epsilon\}$ is finite. Note that

$$\bigwedge_{j=1}^m r_{\alpha+j}^{s(j)} = \prod_{j=1}^m r_{\alpha+j}^{s(j)} = 2^{-m} \prod_{j=1}^m (\mathbf{1} s(j) r_{\alpha+j})$$

(remembering that each $s(j)$ is either $+$ or $-$.) This expands to a sum $\mathbf{1} + \sum_{n=1}^N t_\alpha^n$ where each t_α^n is a family of the form considered above. It follows that the family $\{\alpha \in \mathfrak{a} : \left| \int \left(\bigwedge_{j=1}^m r_{\alpha+j}^{s(j)} - \mathbf{1} \right) (y) d\gamma^\mathfrak{a} \right| \geq \epsilon\}$ is

finite. The transition to the statement (v) (b) proceeds as in the proof of part (iv). $\square$

3. WEAK* SEQUENTIAL CONTINUITY OF LATTICE OPERATIONS.

A number of properties of the dual of a Banach lattice which we will need later, related to the interaction between the lattice operations and the weak* topology in the dual of a Banach lattice, are presented in this section. All these results involve only sequential convergence. The first problem that we address is precisely the question of when the lattice operations in the dual of a Banach lattice are weak* sequentially continuous. Several related conditions are also considered.

The following property was first introduced by van Rooij in [8].

Definition 3.1. A Banach lattice has *property (*)* if, for every sequence (f_n) in X_+^* which converges $\sigma(X^*, X)$ to $f \in X_+^*$ as $n \rightarrow \infty$, we have $|f_n - f| \rightarrow 0$ for $\sigma(X^*, X)$ as $n \rightarrow \infty$.

We record first of all the rather simple fact that property (*) is equivalent to weak* sequential continuity of the lattice operations in the dual.

Lemma 3.2. *A Banach lattice X has property (*) if and only if the lattice operations in X^* are weak* sequentially continuous.*

Proof. It is well known that weak* sequential continuity of the lattice operations is equivalent to weak* sequential continuity of the modulus. If the modulus is weak* sequentially continuous and $f_n \rightarrow f$ for $\sigma(X^*, X)$ then $f_n - f \rightarrow 0$, so that $|f_n - f| \rightarrow 0$ for $\sigma(X^*, X)$ and property (*) holds. Suppose conversely that (*) holds and that $f_n \rightarrow f$ for $\sigma(X^*, X)$. As $||f_n| - |f|| \leq |f_n - f|$ it follows from property (*) that $|f_n - f| \rightarrow 0$ and hence $||f_n| - |f|| \rightarrow 0$ for $\sigma(X^*, X)$. From that it immediately follows that $|f_n| - |f| \rightarrow 0$ and hence that $|f_n| \rightarrow |f|$ for $\sigma(X^*, X)$. $\square$

Two definitions that are at first sight slightly weaker and which might seem easier to verify in practise are the following.

Definition 3.3. A Banach lattice has *property (**)* if, for every sequence (f_n) in X^* such that $f_n \rightarrow f$ and with $|f_n| \rightarrow h \in X^*$ for $\sigma(X^*, X)$ as $n \rightarrow \infty$, we have $h = |f|$.

Definition 3.4. A Banach lattice X has *property (***)* if, for every sequence (f_n) in X^* such that $f_n \rightarrow 0$ and $|f_n| \rightarrow h \in X^*$ for $\sigma(X^*, X)$ as $n \rightarrow \infty$, we have $h = 0$.

These three properties are obviously very close to each other. Exactly how close we now investigate. We separate out some parts of the proof which are of independent interest.

Proposition 3.5. *If Σ is a locally compact Hausdorff space on which every convergent sequence is constant except for a finite number of terms then there is a regular Borel measure μ and Borel measurable functions $f_1, f_2, \dots$ on Σ such that*

- (i) $\mu(\Sigma) = 1$
- (ii) *For all $n \in \mathbb{N}$, $|f_n| = f_1 = \chi_\Sigma$ μ -almost everywhere.*
- (iii) $\int f_m f_n d\mu = 0$ if $m \neq n$.

Proof. Take K to be any compact infinite subset of Σ . By hypothesis, K does not contain any non-trivial convergent sequence. Let (s_n) be a sequence of distinct points of K . We define a sequence (K_n) of infinite compact subsets of K with $K_{2i+1} \cup K_{2i} \subseteq K_i$ and $K_{2i+1} \cap K_{2i} = \emptyset$ for all $i \in \mathbb{N}$. Take $K_1 = K$. As K_1 is compact and (s_n) has no convergent subsequence it has at least two distinct accumulation points, a_2 and a_3 . We can find disjoint compact neighbourhoods K_2 and K_3 of a_2 and a_3 respectively. Each of these contains infinitely many points of the sequence (s_n) . We can repeat the argument in K_2 and find two disjoint neighbourhoods K_4 and K_5 in K_2 of accumulation points of (s_n) . It is clear how the construction proceeds inductively.

For each $i \in \mathbb{N}$, choose $t_i \in K_i$ and then define $g_n \in C_0(\Sigma)^*$ by

$$g_n(x) = 2^{-(n-1)} \sum_{i=2^{n-1}}^{2^n-1} x(t_i)$$

for each $x \in C_0(\Sigma)$. Let $H = \{x \in C_0(\Sigma) : \lim_{n \rightarrow \infty} g_n(x) \text{ exists}\}$. Clearly if $y \in C_0(\Sigma)$ is constantly α on K then $g_n(y) = \alpha$ so that $y \in H$. Any element of $C_0(\Sigma)_+$ is dominated by a function that is constant on K , so we see that H is a majorising subspace of $C_0(\Sigma)$. The functional $g(x) = \lim_{n \rightarrow \infty} g_n(x)$ is a positive linear functional on H . It follows, Corollary 1.5.9 of [7], that g extends to a positive linear functional on $C_0(\Sigma)$, which we also denote by g . Use the Riesz representation theorem to find a positive regular Borel measure μ on Σ such that $g(x) = \int_\Sigma x d\mu$ for all $x \in C_0(\Sigma)$. It is easy to verify that $\mu(\Sigma) = \mu(K) = 1$ and that $\mu(K_i) = 2^{-n+1}$ if $2^{n-1} \leq i \leq 2^n - 1$ and $n \in \mathbb{N}$.

Let $f_1 = \chi_K = \chi_\Sigma$ μ -almost everywhere and also let

$$f_{n+1} = \sum_{i=2^{n-1}}^{2^n-1} (-1)^i \chi_{K_i}$$

so that $|f_{n+1}| = \chi_K = \chi_\Sigma$ μ -almost everywhere and $\int_\Sigma f_m f_n d\mu = 0$ if $m \neq n$. $\square$

Proposition 3.6. *If X is a Banach lattice with an order continuous norm then the lattice operations on X^* are $\sigma(X^*, X)$ -continuous on bounded subsets of X^* .*

Proof. Suppose that we have a bounded net $(f_\gamma)_{\gamma \in \Gamma}$ in X^* which converges $\sigma(X^*, X)$ to f . Pick a real number K such that $\|f_\gamma\|, \|f\| \leq K$,

and hence also $\|f_\gamma\|, \|f\| \leq K$, for all $\gamma \in \Gamma$. Pick any $x \in X$. If $\epsilon > 0$ we can find a finite collection of atoms in X , $\{e_i : i \in F\}$, and a linear combination of these, $x' = \sum_{i \in F} \lambda_i e_i$, such that $\|x - x'\| < \epsilon$. In the dual of the finite dimensional space E spanned by the $\{e_i : i \in F\}$, the $\sigma(E^*, E)$ topology coincides with the norm topology. As $(f_{\gamma|E})$ converges to $f|_E$ for $\sigma(E^*, E)$, it converges in norm and hence $(|f_{\gamma|E}|) = (|f_{\gamma}|_E)$ converges to $|f|_E$ in norm. Thus

$$||f_\gamma| - |f|(x)| \leq |(|f_\gamma| - |f|)(x')| + |(|f_\gamma| - |f|)(x - x')|$$

and we have both $(|f_\gamma| - |f|)(x') \rightarrow 0$ and $|(|f_\gamma| - |f|)(x - x')| \leq 2K\epsilon$ so that $\limsup |(|f_\gamma| - |f|)(x)| \leq 2K\epsilon$. This holds for all $\epsilon > 0$ so we see that $|f_\gamma|(x) \rightarrow |f|(x)$. $\square$

We would like to characterize Banach lattices which are atomic and have an order continuous norm. Later we will give several partial results, but for now let us record a characterisation of Banach lattices for which the dual is atomic.

Theorem 3.7. *For a Banach lattice X the following are equivalent:*

- (i) X^* is atomic.
- (ii) There is no sequence (f_n) in X^* with $0 \neq |f_n| = f \in X_+^*$ for all $n \in \mathbb{N}$ but with $f_n \rightarrow 0$ for $\sigma(X^*, X)$.

Proof. Suppose that X^* is not atomic then there is an $f \in X_+^*$ such that $[0, f]$ does not contain any atoms. Let $J = \{x \in X : f(|x|) = 0\}$ be the null ideal of f . The completion of X/J under the norm $\|x+J\| = f(|x|)$, which is clearly well-defined, is an AL-space which we denote by Y . Since the adjoint of the canonical mapping $\mathcal{J} : X \rightarrow Y$ is one-to-one and identifies the dual of Y with the principal ideal in X^* generated by f , it is easy to see that the dual of Y does not contain any atoms. It follows from the results in section 2 that there is a sequence (g_n) in Y^* with $g_n(y) \rightarrow 0$ for all $y \in Y$ and with $|g_n|$ non-zero and independent of $n \in \mathbb{N}$. As $\mathcal{J}$ is interval preserving, $\mathcal{J}^*$ is a lattice homomorphism so that if $f_n = \mathcal{J}^*(g_n)$ then the sequence $(|f_n|)$ is also constant and for all $x \in X$ we have $f_n(x) = \mathcal{J}^*g_n(x) = g_n(\mathcal{J}x) \rightarrow 0$. This establishes that (ii) implies (i).

Now suppose that X^* is atomic and that there exists a sequence (f_n) in X^* with $f_n \rightarrow 0$ in $\sigma(X^*, X)$ as $n \rightarrow \infty$ but with $|f_n| = f > 0$ for all $n \in \mathbb{N}$. Consider the AL-space Y defined by f as above. As the dual of Y may be identified with the ideal generated by f in X^* , that dual is certainly atomic. Hence the AL-space Y is also atomic. If we let $f_n = \mathcal{J}(g_n)$ then $g_n \rightarrow 0$ in $\sigma(X^*, X)$ whilst $|g_n| = \mathcal{J}^{-1}(f) > 0$ for all $n \in \mathbb{N}$. As the lattice operations in the dual of an atomic AL-space are sequentially continuous for its weak* topology, Theorem 3.12, this is impossible. $\square$

The lattice operations in X^* need not be sequentially $\sigma(X^*, X)$ continuous in such cases.

Example 3.8. Let $X = c$, so that X^* may be identified with ℓ_1 and is certainly atomic. Define $f_n(x) = x_n - x_{n+1}$ for $x = (x_n) \in c$ so that certainly $f_n \rightarrow 0$ for $\sigma(\ell_1, c)$. But $|f_n|(x) = x_n + x_{n+1}$ so that $|f_n| \rightarrow 2\ell \neq 0$, for $\sigma(\ell_1, c)$, where $\ell(x) = \lim_{n \rightarrow \infty} x_n$.

It is useful to note:

Proposition 3.9. *If X is a Banach lattice with an order continuous norm, then X is purely atomic if and only if X^* is purely atomic.*

Proof. This is obvious using Corollary 2.4 and Proposition 2.5. $\square$

Proposition 3.10. *If X is a Banach lattice such that X^* is atomic and H is a closed sublattice of X then H^* is also atomic.*

Proof. If $J : H \rightarrow X$ is the inclusion mapping, which is certainly a lattice homomorphism then, by Theorem 1.4.19 of [7], $J^* : X^* \rightarrow H^*$ is interval preserving. It is a direct consequence of the classical Hahn-Banach theorem that J^* is onto. Because J^* is interval preserving, the image of an atom will be either an atom or zero. Every element of X^* is in the order closure of the linear span of the atoms in X_+^* and J^* is order continuous so every element of H^* is in the order closure of the linear span of the atoms in H_+^* . $\square$

Proposition 3.11. *A Dedekind σ -complete Banach lattice X , for which X^* is atomic, has an order continuous norm and is hence atomic.*

Proof. If X does not have an order continuous norm, then X contains a closed sublattice H which is lattice isomorphic to ℓ_∞ . By Theorem 3.10 H^* , and hence ℓ_∞^* , is atomic. As ℓ_∞^* is not atomic, this contradiction establishes that X does indeed have an order continuous norm. It follows from Theorem 3.9 that X is atomic. $\square$

Theorem 3.12. *Let X be a Banach lattice and consider the following conditions.*

- (i) X is atomic with an order continuous norm.
- (ii) X has property (*).
- (iii) X has property (**).
- (iv) X has property (***)

It is always true that (i) implies (ii) implies (iii) implies (iv). If X is Dedekind σ -complete or is separable or $X = C_0(\Sigma)$, where Σ is a locally compact Hausdorff space, then all four conditions are equivalent.

Proof. That (i) implies (ii) follows from Proposition 3.6. If we assume (ii) and take a sequence (f_n) in X^* with $f_n \rightarrow 0$ and $|f_n| \rightarrow f$ for $\sigma(X^*, X)$, then condition (*) tells us that $||f_n| - f| \rightarrow 0$ for $\sigma(X^*, X)$, as $0 \leq |f_n| \rightarrow f$ for $\sigma(X^*, X)$. Also $0 \leq |f_n| - f_n \rightarrow f$ for $\sigma(X^*, X)$ so

that (*) also shows that $||f_n| - f_n - f| \rightarrow 0$ for $\sigma(X^*, X)$. It follows from the inequalities

$$0 \leq |f_n| \leq ||f_n| - f| + |f_n| - f_n - f \rightarrow 0$$

for $\sigma(X^*, X)$ that $f = 0$. Thus (iii) holds. It is clear that (iii) implies (iv).

Now let us assume that (iv) holds, then certainly condition (ii) of Theorem 3.7 holds so that X^* is atomic. By Theorem 3.9 we need only prove that X has an order continuous norm. If X is Dedekind σ -complete then this follows from Theorem 3.11.

Let us now look at the case that X is separable but does not have an order continuous norm. By Theorem 2.4.2 of [7] we may find $x \in X_+$ and a disjoint sequence (x_n) in $[0, x]$ with $\|x_n\| \geq 1$ for all $n \in \mathbb{N}$. We can now find a bounded sequence in X_+^* with $f_n(x_n) \geq 1$ for all $n \in \mathbb{N}$. As X is separable, on closed bounded subsets of X^* , the compact $\sigma(X^*, X)$ -topology is metrisable. We may thus choose a subsequence such that the sequence (f_n) is $\sigma(X^*, X)$ -convergent to $f \in X_+^*$. The sequence $(|f_n - f|)$ is bounded, so we may (by taking a further subsequence) assume that $|f_n - f| \rightarrow g \in X^*$ for $\sigma(X^*, X)$. By (**), $g = 0$. The process of extracting subsequences does not alter the property that $f_n(x_n) \geq 1$. Note also that

$$0 \leq \sum_{k=1}^n f(x_k) = f\left(\sum_{k=1}^n x_k\right) = f\left(\bigvee_{k=1}^n x_k\right) \leq f(x)$$

for all $n \in \mathbb{N}$ so we certainly have $f(x_n) \rightarrow 0$ as $n \rightarrow \infty$. We hence have

$$1 \leq f_n(x_n) \leq |f_n - f|(x_n) + f(x_n) \leq |f_n - f|(x) + f(x_n) \rightarrow 0$$

as $n \rightarrow \infty$. This contradiction establishes the proof in the case that X is separable.

Finally, we suppose that $X = C_0(\Sigma)$, for some locally compact Hausdorff space Σ . In order to show that X has an order continuous norm we must show that Σ has a discrete topology. This will be accomplished if we can show that every compact subset K of Σ is finite. We split this part of the proof into two cases.

Suppose that K contains a non-trivial convergent sequence $t_n \rightarrow t$. We may suppose that each $t_n \neq t$. Define $f_n, f \in C_0(\Sigma)^*$ by $f_n(x) = x(t_n)$ and $f(x) = x(t)$, where $x \in C_0(\Sigma)$. It is clear that $f_n \rightarrow f$ for $\sigma(C_0(\Sigma)^*, C_0(\Sigma))$ and that $f_n \wedge f = 0$ for all $n \in \mathbb{N}$. Hence $|f - f_n| = f + f_n \rightarrow 2f \neq 0$ for $\sigma(C_0(\Sigma)^*, C_0(\Sigma))$. This contradicts (**).

If K does not contain any non-trivial convergent sequence, then we may invoke Proposition 3.5 to produce a probability measure μ on Σ and an orthonormal sequence (y_n) in $L_2(\mu)$. It follows that $\int_{\Sigma} xy_n d\mu$ and $g(x) = \int_{\Sigma} x d\mu$ then $g_n \rightarrow 0$ for $\sigma(C_0(\Sigma)^*, C_0(\Sigma))$ whilst $|g_n| = g \neq 0$ for all $n \in \mathbb{N}$. This contradicts (***) so the proof is complete. $\square$

Open Problem 3.13. Are the four conditions of Theorem 3.12 equivalent in general?

We can use the non-sequential analogue of property (***) to characterise atomic Banach lattice with an order continuous norm.

Theorem 3.14. *The following conditions on a Banach lattice X are equivalent:*

- (i) X is atomic with an order continuous norm.
- (ii) If bounded net (f_γ) in X^* is $\sigma(X^*, X)$ convergent to 0 and $|f_\gamma| \rightarrow f \in X^*$ $\sigma(X^*, X)$ then $|f_\gamma| \rightarrow 0$ $\sigma(X^*, X)$.

Proof. That (i) implies (ii) is clear. If we suppose that (ii) holds then certainly Theorem 3.7 tells us that X^* is atomic. If we can show that X has an order continuous norm then Theorem 3.9 will tell us that X is also atomic and we will have established (i). If X does not have an order continuous norm then by Theorem 2.4.2 of [7] there is an order interval in X containing a disjoint sequence which does not converge to 0 in norm. By taking a subsequence and scaling we can assume that the disjoint sequence (x_n) lies in the order interval $[0, x]$ and that $\|x_n\| \geq 1$ for all $n \in \mathbb{N}$. If $f \in X_+^*$ then as $0 \leq \sum_{k=1}^n f(x_k) = f(\sum_{k=1}^n x_k) \leq f(x)$ we see that $f(x_k) \rightarrow 0$. By Proposition 2.3.4 (ii) of [7] there is a disjoint sequence (f_n) in the positive part of the unit ball of X^* such that $\limsup f_n(x_n) = \limsup \|f_n\|$ and by taking a subsequence we may suppose that $f_n(x_n) \geq 1$ for all $n \in \mathbb{N}$.

There is a subnet (f_γ) of the sequence (f_n) converging $\sigma(X^*, X)$ to some limit $f \in X^*$, using $\sigma(X^*, X)$ -compactness of the unit ball in X^* . By taking a further subnet (f_δ) we may also require that $|f_\delta - f| \rightarrow g \in X^*$ whilst still having $f_\delta - f \rightarrow 0$. Condition (ii) forces $|f_\delta - f| \rightarrow 0$. In particular, 0 is a cluster point of the sequence $(|f_n - f|(x))$ and we may take a subsequence with $|f_{n_k} - f|(x) \rightarrow 0$. Thus

$$\begin{aligned} 1 \leq f_{n_k}(x_{n_k}) &\leq |f_{n_k} - f|(x_{n_k}) + f(x_{n_k}) \\ &\leq |f_{n_k} - f|(x) + f(y_{n_k}) \rightarrow 0 \end{aligned}$$

which contradiction completes the proof. $\square$

4. WHEN DO THE REGULAR OPERATORS FORM A LATTICE?

Although I have talked in the section title about *regular* operators, it turns out that in most cases where we can show that the regular operators form a lattice we will in fact have shown that the (potentially) larger class of order bounded operators forms a lattice (and therefore coincides with the regular operators.) We will therefore define both classes.

Proposition 4.1. *Let X and Y be Archimedean vector lattices and $T : X \rightarrow Y$ be a linear operator. The following conditions are equivalent:*

- (i) For every $x \in X_+$ there is $y \in Y_+$ such that $T([0, x]) \subset [-y, y]$.
- (ii) For every $x \in X_+$ there is $y \in Y_+$ such that $T([-x, x]) \subset [-y, y]$.
- (iii) For every order bounded set $A \subset X$, $T(A)$ is order bounded in Y .

Proof. If (i) holds and $x \in x_+$ then there is $y \in Y_+$ such that $T([0, 2x]) \subset [-y, y]$. Hence $T([-x, x]) = T([0, 2x]) - T(x) \subset [-y, y] - T(x)$. If $z = y + |T(x)|$ and $t \in [-y, y]$ then $|t - T(x)| \leq |t| + |T(x)| \leq y + |T(x)|$ so that $T([-x, x]) \subset [-z, z]$ and (ii) holds. If (ii) holds and A is order bounded, let $x \in X_+$ be such that $A \subset [-x, x]$. By (ii) there is $y \in Y_+$ such that $T([-x, x]) \subset [-y, y]$ and hence $T(A) \subset [-y, y]$ so that $T(A)$ is order bounded. Finally, that (iii) implies (i) is immediate from $[0, x]$ being order bounded. $\square$

Definition 4.2. If X and Y are vector lattices and $T : X \rightarrow Y$ is a linear operator then:

- (i) If T satisfies the equivalent conditions of Proposition 4.1 then it is termed *order bounded*.
- (ii) If $T(X_+) \subseteq Y_+$ then T is *positive*.
- (iii) If there are positive operators $U, V : X \rightarrow Y$ such that $T = U - V$ then T is *regular*.

It should be clear that positive linear operators preserve all inequalities, as $x \geq y \Rightarrow x - y \geq 0 \Rightarrow T(x) - T(y) = T(x - y) \geq 0 \Rightarrow T(x) \geq T(y)$. It will be equally clear that the positive operators do not form a vector space as, for example, multiplying a positive operator by a negative number will destroy positivity (unless the operator is the zero operator). The regular operators are precisely the linear span of the positive operators. Note that T is regular if and only if there is U such that $U \geq T, 0$ (write $T = U - (U - T)$). It is also true that the order bounded operators form a vector space.

Proposition 4.3. If X and Y are Archimedean vector spaces then the set of all order bounded linear operators form a vector space under pointwise addition and scalar multiplication.

Proof. If S and T are two order bounded operators from X into Y , and $x \in X_+$ then there are $y_S, y_T \in Y_+$ such that $S([0, x]) \subset [-y_S, y_S]$ and $T([0, x]) \subset [-y_T, y_T]$ and hence

$$\begin{aligned} (S + T)([0, x]) &\subset S([0, x]) + T([0, x]) \\ &\subset [-y_S, y_S] + [-y_T, y_T] \\ &\subset [-(y_S + y_T), (y_S + y_T)] \end{aligned}$$

so that $S + T$ is order bounded. If, also, $\lambda \in \mathbb{R}$ then

$$(\lambda S)([0, x]) = \lambda S([0, x]) \subset \lambda[-y_S, y_S] = [-|\lambda|y_S, |\lambda|y_S]$$

so that λS is also order bounded. $\square$

Definition 4.4. If X and Y are Archimedean vector lattices then the vector space of all order bounded operators from X into Y is denoted by $\mathcal{L}^b(X, Y)$ and the vector space of regular operators from X into Y by $\mathcal{L}^r(X, Y)$.

It will be clear that regular operators are order bounded and that if the space of order bounded operators forms a lattice then in fact every order bounded operators is regular. Furthermore, if X is a Banach lattice and Y a normed lattice then every order bounded operator from X into Y is norm bounded. There are many well known examples to show that norm bounded operators need not be regular or even order bounded. Less well know are explicit examples of order bounded operators that are not regular.

Example 4.5. *There are Archimedean vector lattices X and Y such that not every order bounded operator from X into Y is regular.*

Proof. We take

$$X = \{(x_m) : x_m \in c \forall m \in \mathbb{N} \text{ and } \exists \alpha \in \mathbb{R} \text{ with } \lim_{m \rightarrow \infty} \|x_m - (\alpha)\|_\infty = 0\},$$

where (α) denotes the constantly α sequence and $Y = c(\mathbb{N} \times \mathbb{N})$. We denote by $x_{m,n}$ the n 'th term in x_m and let $x_{m,\infty} = \lim_{n \rightarrow \infty} x_{m,n}$. We define $T : X \rightarrow Y$ by $T((x_m)) = (x_{m,n+1} - x_{m,n})$. This is clearly a linear operator. To see that it actually maps X into Y , take $(x_m) \in X$ and let α be such that $\|x_m - (\alpha)\|_\infty \rightarrow 0$. If $\epsilon > 0$ then there is m_0 such that $\|x_m - (\alpha)\|_\infty < \epsilon/2$ for all $m \geq m_0$. Then we have, for $m > m_0$ and for all $n \in \mathbb{N}$,

$$\begin{aligned} |x_{m,n+1} - x_{m,n}| &\leq |x_{m,n+1} - \alpha| + |\alpha - x_{m,n}| \\ &\leq \|x_m - (\alpha)\|_\infty + \|(\alpha) - x_m\|_\infty < \epsilon. \end{aligned}$$

For each $m \leq m_0$ we have $x_m \in c$ so there are $\ell_m \in \mathbb{R}$ and $n_0(m) \in \mathbb{N}$ such that $n > n_0(m) \Rightarrow |x_{m,n} - \ell_m| < \epsilon/2$ and hence $|x_{m,n+1} - x_{m,n}| \leq |x_{m,n+1} - \ell_m| + |\ell_m - x_{m,n}| < \epsilon$. Thus if $|x_{m,n+1} - x_{m,n}| \geq \epsilon$ then $m \leq m_0$ and $n \leq n_0(m)$ (a finite number of possibilities) so that (taking $\alpha = 0$) we have $T((x_m)) \in Y$.

Note also that T is order bounded, for any order bounded set is contained in an order interval $[-\mathbf{1}, \mathbf{1}]$, where we let $\mathbf{1}$ denote the sequence all of whose entries are the constantly one sequence. If x lies in this interval then $|x_{m,n}|, |x_{m,n+1}| \leq 1$ so that $|x_{m,n} - x_{m,n+1}| \leq 2$ and it follows that $T((x_m)) \in [-2 \times \mathbf{1}, 2 \times \mathbf{1}]$. Of course, $2 \times \mathbf{1}$ means twice the constantly one function, so is the constantly two function.

Let e_m denote the element of X which has all its entries the zero sequence, except for the m 'th which is the constantly one sequence and $e_{m,n}$ be that element with the only non-zero entry the m 'th which

in turn has all entries 0 except the n 'th which is 1. Then $\mathbf{e}_m \geq \mathbf{e}_{m,n}$. If $U \geq T, 0$ then

$$U(\mathbf{e}_m) \geq T(\mathbf{e}_m) \geq T(\mathbf{e}_{m,n})$$

so that the (m, n) 'th entry in $U(\mathbf{e}_m)$ is at least 1 for all $n \in \mathbb{N}$, so for infinitely many n . It follows that the limit of $U(\mathbf{e}_m)$ is at least 1, and hence $U(\sum_{m=1}^M \mathbf{e}_m) = \sum_{m=1}^M U(\mathbf{e}_m)$ has limit at least M . As $\mathbf{1} \geq \sum_{m=1}^M \mathbf{e}_m$ we have $U(\mathbf{1}) \geq \sum_{m=1}^M U(\mathbf{e}_m) \geq M$ for all $M \in \mathbb{N}$. This is clearly impossible so that such a U cannot exist and hence T is not regular. $\square$

Although the space of all regular operators between two Banach lattices is a positively generated ordered normed space it is not, in general, a lattice.

Example 4.6. $\mathcal{L}^r(X, Y)$ need not be a lattice.

Proof. Consider the two operators $T, U : L_1([0, 1]) \rightarrow c$ defined by $Ux = \|x\|_1 \mathbf{1}$ and $Tx = \sum_{n=1}^{\infty} r_n(x) e_{2n}$. As $r_n(x) \rightarrow 0$ we actually have $Tx \in c_0$. It is clear that $U \geq T, 0$ so that T is regular. Suppose that S were the supremum in $\mathcal{L}^r(L_1([0, 1]), c)$ of t and 0. For each $k \in \mathbb{N}$ let P_k denote the band projection of c onto the one-dimensional band generated by e_k . It is clear that if k is odd then $(I - P_k) \circ U \geq T, 0$. It follows that for odd k we have

$$(I - P_k) \circ U \chi_{[0,1]} \geq S \chi_{[0,1]} \geq 0.$$

It follows that the k 'th entry in $S \chi_{[0,1]}$ is zero. On the other hand, as $S \chi_{[0,1]} \geq S r_n = e_{2n}$ the even entries in $S \chi_{[0,1]}$ are at least 1. We therefore see that $S \chi_{[0,1]} \notin c$ and thus $\mathcal{L}^r(L_1([0, 1]), c)$ is not a lattice. $\square$

In all known examples of vector lattices of operators the lattice operations satisfy the so-called Freudenthal-Kantorovich-Riesz formulae. These are the formulae $T^+(x) = \sup T([0, x])$, $T^-(x) = \sup T([-x, 0])$, $|T|(x) = \sup T([-x, x])$,

$$(S \vee T)(x) = \sup \{S(y) + T(x) : y, z \in X_+ \text{ and } x = y + z\}$$

and

$$(S \wedge T)(x) = \inf \{S(y) + T(x) : y, z \in X_+ \text{ and } x = y + z\}.$$

In fact, no example is known to the author of a single operator which has a modulus which is not given by the Freudenthal-Kantorovich-Riesz formula.

It is also not the case that $\mathcal{L}^r(X, Y)$ is complete under the operator norm, in general. There is, however, another norm that we can consider on it.

Definition 4.7. If X and Y are normed lattices and $T \in \mathcal{L}^r(X, Y)$ then

$$\|T\|_r = \inf\{\|S\| : S \geq \pm T, S \in \mathcal{L}^r(X, Y)\}.$$

Proposition 4.8. For all normed lattices X and Y , $\|\cdot\|_r$ is a norm on $\mathcal{L}^r(X, Y)$, and $\|T\| \leq \|T\|_r$ for all $T \in \mathcal{L}^r(X, Y)$. If Y is a Banach lattice then $\mathcal{L}^r(X, Y)$ is a Banach space under $\|\cdot\|_r$.

Proof. It is routine to show that $\|\cdot\|_r$ is a norm and that $\|T\| \leq \|T\|_r$ for all $T \in \mathcal{L}^r(X, Y)$. Let us now assume that Y is norm complete and show that $\mathcal{L}^r(X, Y)$ is norm complete under $\|\cdot\|_r$. Let (T_n) be a $\|\cdot\|_r$ -Cauchy sequence in $\mathcal{L}^r(X, Y)$. By passing to a subsequence we may assume that

$$\|T_n - T_{n+1}\|_r < 2^{-n} \text{ for all } n \in \mathbb{N}$$

and we still need only prove convergence to some limit $T \in \mathcal{L}^r(X, Y)$. By the definition of $\|\cdot\|_r$ we may find $S_n \in \mathcal{L}^r(X, Y)$ with

$$\|S_n\|_r < 2^{-n} \text{ and } S_n \geq \pm(T_n - T_{n+1}).$$

Since $\|T_n - T_{n+1}\| \leq \|T_n - T_{n+1}\|_r$, (T_n) is a Cauchy sequence in the Banach space $\mathcal{L}(X, Y)$ (with the operator norm, $\|\cdot\|$), so converges in the operator norm to some $T \in \mathcal{L}(X, Y)$. For each $m \in \mathbb{N}$, the series

$$\sum_{n=m}^{\infty} S_n$$

is convergent in the operator norm to $Q_n \in \mathcal{L}(X, Y)_+$ with

$$\|Q_n\| \leq \sum_{n=m}^{\infty} \|S_n\| \leq \sum_{n=m}^{\infty} 2^{-n} \leq 2^{1-n}.$$

If $x \in X_+$ then

$$\begin{aligned} (T_n - T)x &= \lim_{m \rightarrow \infty} (T_n - T_m)x \\ &= \sum_{m=n}^{\infty} (T_m - T_{m+1})x \\ &\leq \sum_{m=n}^{\infty} S_m x = Q_n x \end{aligned}$$

so that $T_n - T \leq Q_n$. Similarly $T - T_n \leq Q_n$, so that T is regular and $\|T_n - T\|_r \leq \|Q_n\| < 2^{1-n}$ so that $T_n \rightarrow T$ for $\|\cdot\|_r$. $\square$

We refer to $\|\cdot\|_r$ in future as the *regular norm*. The operator and regular norms are not, in general, even equivalent on the space of regular operators.

We start our investigation by looking at those domains and ranges which always work. We look first at the nicest possible domains. The

following result is due to van Rooij, see [8], although we present the proof given by Chen in [3].

Theorem 4.9. *The following conditions on a Banach lattice X are equivalent:*

- (i) X is atomic with an order continuous norm.
- (ii) For every Banach lattice Y , $\mathcal{L}^b(X, Y) = \mathcal{L}^r(X, Y)$ is a Banach lattice under the regular norm in which the lattice operations are given by the Freudenthal-Kantorovich-Riesz formulae.
- (iii) For every compact Hausdorff space K , $\mathcal{L}^r(X, C(K))$ is a vector lattice.

Proof. We start by establishing that (i) implies (ii). If X is atomic with an order continuous norm, let Ω be the set of all atoms of norm 1, so that for each $x \in X$ there is a function $\lambda : \Omega \rightarrow \mathbb{R}$ such that $x = \sum_{\omega \in \Omega} \lambda(\omega)\omega$ with the sum being norm convergent and therefore with only countably many non-zero terms. Each $T \in \mathcal{L}^b(X, Y)$ is completely specified by knowing each $T\omega$ and clearly (by continuity) $Tx = \sum_{\omega \in \Omega} \lambda(\omega)T\omega$. As $T \in \mathcal{L}^b(X, Y)$, $\mathcal{J}_Y \circ T \in \mathcal{L}^b(X, Y^{**}) = \mathcal{L}^r(X, Y^{**})$ so we may form $U = |\mathcal{J}_Y \circ T|$ and again $Ux = \sum_{\omega \in \Omega} \lambda(\omega)U\omega$. It is clear that $U\omega = |\mathcal{J}_Y \circ T\omega|$. It follows that the sum $\sum_{\omega \in \Omega} \lambda(\omega)|T\omega|$ converges in Y and it is clearly the modulus of T in $\mathcal{L}^b(X, Y)$. This proves the equality of $\mathcal{L}^r(X, Y)$ with $\mathcal{L}^b(X, Y)$ and that this space is a lattice. The formula for the modulus makes it clear that the Freudenthal-Kantorovich-Riesz formulae hold. It is clear that (ii) implies (iii).

Now suppose that (iii) holds and let $(f_\gamma)_{\gamma \in \Gamma}$ be a bounded net in X^* with $f_\gamma \rightarrow 0$ for $\sigma(X^*, X)$ and $|f_\gamma| \rightarrow f \in X^*$ for $\sigma(X^*, X)$. We show that $f = 0$ and then invoke Theorem 3.14 to establish (i). Take $K = \Gamma \times \{1, 2\} \cup \{\infty\}$, where $\infty \notin \Gamma \times \{1, 2\}$. Define a topology on K by specifying that U is open if $\infty \notin U$ or if there is $\gamma_0 \in \Gamma$ such that $(\gamma, i) \in U$ for all $\gamma \geq \gamma_0$ and for $i = 1, 2$. For this topology all points except ∞ are isolated and the nets $((\gamma, 1))_{\gamma \in \Gamma}$ and $((\gamma, 2))_{\gamma \in \Gamma}$ both converge to ∞ . The space of all bounded continuous real valued functions on K , $C_b(K)$, with the pointwise partial order and supremum norm, is a unital AM-space. Thus $\mathcal{L}^r(X, C_b(K))$ is a vector lattice.

Define $T, U : X \rightarrow C_b(K)$ by, for $x \in X$,

$$\begin{aligned} Tx(\gamma, 1) &= f_\gamma(x) & Ux(\gamma, 1) &= |f_\gamma|(x) \\ Tx(\gamma, 2) &= 0 & Ux(\gamma, 2) &= f(x) \\ Tx(\infty) &= 0 & Ux(\infty) &= f(x). \end{aligned}$$

Clearly $U \geq \pm T$ so that $T \in \mathcal{L}^r(X, C_b(K))$ and hence $|T|$ exists. For each $\gamma \in \Gamma$ the function h_γ on K defined by

$$\begin{aligned} h_\gamma(\beta, 1) &= 1 & \text{for all } \beta \in \Gamma \\ h_\gamma(\infty) &= 1 \\ h_\gamma(\beta, 2) &= 1 & \text{if } \beta \not< \gamma \\ h_\gamma(\beta, 2) &= 0 & \text{if } \beta < \gamma \end{aligned}$$

lies in $C_b(K)$. It is clear that $h_\gamma \times |T| \geq \pm T$ and therefore if $x \in X_+$ we have $|T|x(\beta, 2) = 0$ for all $\beta < \gamma$. This holds for all γ so by continuity $|T|x(\infty) = 0$. On the other hand, if $x \geq 0$ then $|T|x(\gamma, 1) \geq \bigvee \{Ty(\gamma, 1) : |y| \leq x\} = \bigvee \{f_\gamma(y) : |y| \leq x\} = |f_\gamma|(x)$ and the continuity of $|T|x$ shows that $|T|x(\gamma, 1) \rightarrow |T|x(\infty) = 0$ and hence $|f_\gamma|(x) \rightarrow 0$. This holds for all $x \in X_+$ and therefore for all $x \in X$, so that $f = 0$ as desired and the proof is complete. $\square$

The positive part of the result at the other extreme is best proved in a general setting which we can apply in various places. The arguments used are familiar and we will not go into detail.

Theorem 4.10. *Let X and Y be Archimedean vector lattices and $\mathcal{L}$ is equal to either $\mathcal{L}^r(X, Y)$ or to $\mathcal{L}^b(X, Y)$ then the following are equivalent:*

- (i) *For all $x \in X_+$ and $S \in \mathcal{L}$ the set $S([0, x])$ has a supremum in Y .*
- (ii) *For all $x \in X_+$ and $S \in \mathcal{L}$ the set $S([-x, 0])$ has a supremum in Y .*
- (iii) *For all $x \in X_+$ and $S \in \mathcal{L}$ the set $S[-x, x]$ has a supremum in Y .*

If these statements do hold then $\mathcal{L}$ is a vector lattice in which the lattice operations are given by the Freudenthal-Kantorovich-Riesz formulae. In particular, if $\mathcal{L} = \mathcal{L}^b(X, Y)$ then we have $\mathcal{L}^b(X, Y) = \mathcal{L}^r(X, Y)$.

Proof. The equivalence of the statements about the existence of suprema is easy. If these suprema do exist, define $U : X_+ \rightarrow Y$ by $U(x) = \sup S([0, x])$. It is clear that $U(\lambda x) = \lambda U(x)$ for all $x \in X_+$ and $\lambda \geq 0$. The Riesz Decomposition Property tells us that if $x, y \in X_+$ then

$[0, x + y] = [0, x] + [0, y]$ so that

$$\begin{aligned}
U(x + y) &= \sup S([0, x + y]) \\
&= \sup S([0, x] + [0, y]) \\
&= \sup (S([0, x]) + S([0, y])) \\
&= \sup S([0, x]) + \sup S([0, y]) \\
&= U(x) + U(y).
\end{aligned}$$

Now U has a linear extension to the whole of X which we also denote by U . Note that as U is positive, $U \in \mathcal{L}$ and it can be shown that U is the supremum in $\mathcal{L}$ of S and 0. The rest of the Freudenthal-Kantorovich-Riesz formulae are easily deduced. $\square$

At the other extreme from van Rooij's result we have:

Theorem 4.11. *The following conditions on a Banach lattice Y are equivalent:*

- (i) Y is Dedekind complete.
- (ii) For every Banach lattice X , $\mathcal{L}^b(X, Y)$ is a Banach lattice under the regular norm in which all lattice operations satisfy the Riesz-Kantorovich formulae.
- (iii) For every Banach lattice X , $\mathcal{L}^r(X, Y)$ is a vector lattice.

Proof. That (i) implies that $\mathcal{L}^b(X, Y)$ is a vector lattice follows from the last theorem, once we realise that because $\mathcal{L}^b(X, Y) = \mathcal{L}^r(X, Y)$ is a vector lattice the regular norm satisfies $\|T\|_r = \||T|\|$. If $S, T \in \mathcal{L}^r(X, Y)$ and $|S| \leq |T|$ then we have

$$\|S\|_r = \||S|\| \leq \||T|\| = \|T\|_r$$

so that the Banach space $\mathcal{L}^b(X, Y)$ is a Banach lattice. Clearly (ii) implies (iii).

In order to prove that (iii) implies (i), since Y is certainly uniformly complete, in order to prove that it is Dedekind complete it suffices, by Theorem 1.10, to prove that any disjoint family of positive elements $(y_i)_{i \in I} \in Y$, which has an upper bound $y \in Y$, must have a supremum. Let $c(I)$ denote the space of all real-valued functions f on I with the property that there is a real α such that for all $\epsilon > 0$ the set $\{i \in I : |f(i) - \alpha| > \epsilon\}$ is finite. We write ℓ_f for this (unique) real α . Under the supremum norm and pointwise partial order $c(I)$ is a Banach lattice. Define an operator $T : c(I) \rightarrow Y$ by

$$T(f) = \sum_{i \in I} (f(i) - \ell_f) y_i.$$

This series has only countably many non-zero terms and is Cauchy because if $\epsilon > 0$ and $F = \{i \in I : |f(i) - \ell_f| > \epsilon\}$ then for any finite

set $G \subset \mathbb{N} \setminus F$ we have

$$\begin{aligned}
\left| \sum_{i \in G} (f(i) - \ell_f) y_i \right| &\leq \sum_{i \in G} |f(i) - \ell_f| y_i \\
&\leq \epsilon \sum_{i \in G} y_i \\
&= \epsilon \bigvee_{i \in G} y_i \\
&\leq \epsilon y
\end{aligned}$$

so that $\left\| \sum_{i \in G} (f(i) - \ell_f) y_i \right\| \leq \epsilon \|y\|$. As Y is norm complete, the series converges. Let us also define $U : c(I) \rightarrow Y$ by

$$U(f) = T(f) + \ell_f y.$$

If $f \geq 0$ then $(U - T)(f) = \ell_f y \geq 0$ so that $U \geq T$. Also, for any $f \in c(I)_+$ and $i \in I$ we have $\ell_f \geq \ell_f - f(i), 0$ so that

$$\ell_f y \geq \ell_f y_i \geq (\ell_f - f(i)) y_i.$$

As the (y_i) are disjoint, we must have $\ell_f y \geq \sum_{i \in I} (\ell_f - f(i)) y_i$. I.e. $(U - T)(f) \geq -T(f)$ so that $U - T \geq -T$ and hence $U \geq 0$. It follows T is regular and therefore has a positive part, T^+ .

Writing $\mathbf{1}_I$ for the constantly one function in $c(I)$, we claim that $T^+(\mathbf{1}_I)$ is the supremum in Y of the family (y_i) , which will complete the proof. Firstly note that $T(\mathbf{1}_I)$ is an upper bound for the (y_i) as if we let e_i denote the function that takes the value 1 at i and 0 on the rest of I , then $\mathbf{1}_I \geq e_i \geq 0$ so that

$$T^+(\mathbf{1}_I) \geq T^+(e_i) \geq T(e_i) = y_i.$$

On the other hand the previous paragraph constructed a positive majorant U for T with $U(\mathbf{1}_I) = y$. Since $U \geq T^+$, we must have $y \geq U(\mathbf{1}_I) \geq T^+(\mathbf{1}_I)$. As y was any upper bound for the family (y_i) , we see that $T^+(\mathbf{1}_I)$ is indeed the supremum of the family (y_i) . $\square$

Note that we are not assuming, in the proof of (iii) implies (i), that the lattice operations are given by the Freudenthal-Kantorovich-Riesz formulae.

The Dedekind completeness of the range space can be weakened if the domain is restricted in size. We present here several versions of such results. The important extra feature here is that we need only take a single (appropriate) domain space rather than a whole class of them. The results involving property (*) are all due to van Rooij [8].

Theorem 4.12. *For a Banach lattice X the following are equivalent:*

- (i) X has property (*).
- (ii) $\mathcal{L}^r(X, c)$ is a vector lattice.

Proof. Suppose that (i) holds and that $T, U \in \mathcal{L}^r(X, c)$ with $U \geq T, 0$. For each $n \in \mathbb{N}$ define two sequences of functionals in X^* by $f_n(x) = (Tx)_n$ and $g_n(x) = (Ux)_n$ and define also another two functionals by $f(x) = \lim_{n \rightarrow \infty} Tx$ and $g(x) = \lim_{n \rightarrow \infty} |T|x$, so that $g, g - f, g_n, g_n - f_n \geq 0$. In the weak* topology we have $\lim_{n \rightarrow \infty} (g_n - f_n) = g - f$, so property (*) tells us that $\lim_{n \rightarrow \infty} |g_n - f_n - g + f| = 0$. As we also have $\lim_{n \rightarrow \infty} g_n = g$ so that $\lim_{n \rightarrow \infty} |g_n - g| = 0$, we see that $\lim_{n \rightarrow \infty} |f_n - f| = 0$. As $||f_n| - |f|| \leq |f_n - f|$ it follows that $\lim_{n \rightarrow \infty} |f_n| = |f|$. We may therefore define $S : X \rightarrow c$ by $Sx = (|f_n|(x))$ and it is clear that $S = T^+$.

If (ii) holds then take $f, f_1, f_2, \dots \in X^*$ with $f_n \rightarrow f$ for the weak* topology. Define $T : X \rightarrow c$ by

$$\begin{aligned} (Tx)_{2n} &= (f - f_n)(x) \\ (Tx)_{2n-1} &= 0. \end{aligned}$$

If $x \in X_+$ then $Tx \leq f(x)\mathbf{1}$ so that $T \in \mathcal{L}^r(X, c)$ and therefore $|T|$ exists. By the same argument as used in Example 4.6 $(|T|x)_{2n} = |f - f_n|(x)$ whilst $(|T|x)_{2n-1} = 0$. Therefore $\lim_{n \rightarrow \infty} |f - f_n|(x) = 0$ for all $x \in X$ as required. $\square$

Condition (*) is a very restrictive one, but one that is far more important in our considerations than the special nature of c might imply.

Theorem 4.13. *If X is a Banach lattice and there is a Banach lattice Y which is not Dedekind σ -complete but with $\mathcal{L}^r(X, Y)$ being a vector lattice then X has property (*).*

Proof. We will suppose that X has property (*) and that $\mathcal{L}^r(X, Y)$ is a vector lattice and show that Y is Dedekind σ -complete. It is easy to see that for every ideal Y_0 in Y , $\mathcal{L}^r(X, Y_0)$ is also a vector lattice. If Y were not Dedekind σ -complete there would be a principal ideal in Y which was not Dedekind σ -complete, so we may assume that $Y = C(K)$ for some compact Hausdorff space. In order to show that $C(K)$ is Dedekind σ -complete it suffices by Theorem 1.9 (or Proposition 2.1.5 of [7]) to show that any disjoint sequence (h_n) in $C(K)_+$ which is bounded above by χ_K has a supremum.

As X does not have property (*) we can find $f_n, f \in X_+^*$ with $f_n \rightarrow f$ weak* yet such that there is $x_0 \in X_+$ with $|f - f_n|(x_0) \not\rightarrow 0$. By extracting a subsequence we can ensure that this last sequence is bounded away from 0 and hence that $|f - f_n|(x_0) \geq 1$ for all $n \in \mathbb{N}$. Now set $s_n = |f - f_n|(x_0)$ and set $g_n = s_n^{-1}f_n + (1 - s_n^{-1})f$. We still have $g_n \rightarrow f$ weak* and now $|g_n - f|(x_0) = 1$ for all $n \in \mathbb{N}$.

If we define $T : X \rightarrow C(K)$ by

$$Tx = \sum_{n=1}^{\infty} (f(x) - g_n(x))h_n$$

noting that the sequence is uniformly convergent. As $Tx \leq f(x)\chi_K$ for all $x \geq 0$. T is regular and therefore has a modulus $|T|$. We claim that $|T|(x_0) = \sup\{f_n : n \in \mathbb{N}\}$.

Pick $m \in \mathbb{N}$ and let B_m denote the operator of pointwise multiplication by h_m on $C(K)$, then for each $x \in X$ we have $(B_m T)(x) = (f - f_m)(x) \times h_m^2$ so that $h_m |T|(x_0) = (B_m |T|)(x_0) = |B_m T|(x_0) = |f - g_m|(x_0) \times h_m^2 = h_m^2$. Thus $|T|(x_0)$ coincides with h_m on the set $\{k \in K : h_m(k) > 0\}$. Similarly, if $h \in C(K)_+$ and $hh_n = 0$ for all $n \in \mathbb{N}$ then $h \times |T|(x_0) = 0$. It follows that the continuous function coincides on a dense subset of K with the pointwise supremum of the (h_n) so must be their supremum in $C(K)$. $\square$

Recall that under several quite weak conditions (see Theorem 3.12), property (*) implies that X is atomic with an order continuous norm. Note that X being atomic with order continuous norm is enough, by Theorem 4.9, to guarantee that $\mathcal{L}^r(X, Y)$ is a vector lattice. This points out that there is actually quite a small gap in our knowledge here.

Corollary 4.14. *If X is a separable Banach lattice and Y is any Banach lattice the following are equivalent:*

- (i) *Either X is atomic with an order continuous norm or Y is Dedekind σ -complete.*
- (ii) *$\mathcal{L}^b(X, Y)$ is a Banach lattice under the regular norm in which all lattice operations satisfy the Riesz-Kantorovich formulae.*
- (iii) *$\mathcal{L}^r(X, Y)$ is a Banach lattice under the regular norm.*
- (iv) *$\mathcal{L}^r(X, Y)$ is a vector lattice.*

Proof. If Y is Dedekind σ -complete and T is order bounded and $x \in X_+$ then we may find a countable dense subset $A \subseteq [-x, x]$. The set $T([-x, x])$ is bounded above in F since T is order bounded, so that the countable set $T(A)$ is bounded above and has a supremum, Sx . The density of A in $[-x, x]$ and the continuity of T means that Sx is also the supremum of the set $T([-x, x])$. Now Theorem 4.10 shows that $\mathcal{L}^b(X, Y) = \mathcal{L}^r(X, Y)$ is a vector lattice. We have already seen the proof that we actually have a Banach lattice.

If X is atomic with an order continuous norm then the conclusion (ii) follows from Theorem 4.9. In either case we have established that (i) implies (ii).

It is clear that (ii) implies (iii) and also that (iii) implies (iv). It follows from Theorem 4.13 that either Y is Dedekind σ -complete or X has property (*). But in the latter case, by Theorem 3.12, the fact that X is assumed to be separable forces it to be atomic with an order continuous norm. $\square$

We can similarly obtain a complete answer to the question of when $\mathcal{L}^r(X, Y)$ is a vector lattice when we add the assumption that X has an order continuous norm.

Theorem 4.15. *Let X be a Banach lattice with an order continuous norm and Y be any Banach lattice. The following are equivalent:*

- (i) *Either X is atomic with an order continuous norm or Y is Dedekind $<\mathfrak{a}$ -complete where $\mathfrak{a}$ is the smallest cardinal that is greater than the density character of every order interval in X .*
- (ii) *$\mathcal{L}^b(X, Y)$ is a Banach lattice under the regular norm in which all lattice operations satisfy the Freudenthal-Kantorovich-Riesz formulae.*
- (iii) *$\mathcal{L}^r(X, Y)$ is a vector lattice.*

Proof. The proof that (i) implies (ii) uses either Theorem 4.13 or an obvious modification of part of the proof of Corollary 4.14. Clearly (ii) implies (iii).

Now assume (iii). If X is not atomic choose $\mathfrak{b} < \mathfrak{a}$ such that X contains a homogenous band, B , with order intervals all having density character $\mathfrak{b}$, by Theorem 2.4. If $0 \neq e \in B_+$ and $\phi \in B_+$ is an e -integral, by Theorem 2.7 there is a (ϕ, e) -Rademacher system $\{(e_\beta, \phi_\beta) : \beta \in \mathfrak{b}\}$. We show that Y must be $\mathfrak{b}$ -complete, by proving that every order bounded disjoint family $(y_\beta)_{\beta \in \mathfrak{b}}$ in Y_+ has a supremum, using Theorem 1.9. Let y be an upper bound for this family.

Define operators $S, T : X \rightarrow Y$ by $S(x) = \sum_{\beta \in \mathfrak{b}} \phi_\beta(x) y_\beta$ and $T(x) = \phi(x)y$. The series for S converges because for each $x \in X$ and each $\epsilon > 0$ the set $\{\beta \in \mathfrak{b} : |\phi_\beta(x)| \geq \epsilon\}$ is finite and because of the order boundedness of the family $(y_\beta)_{\beta \in \mathfrak{b}}$. If $x \in X_+$ and F is a finite subset of $\mathfrak{b}$ then we have

$$\begin{aligned} T(x) - \sum_{\beta \in F} \phi_\beta(x) y_\beta &= \phi(x)y - \sum_{\beta \in F} \phi_\beta(x) y_\beta \\ &\geq \phi(x)y - \sum_{\beta \in F} \phi(x) y_\beta \\ &= \phi(x)y - \phi(x) \sum_{\beta \in F} y_\beta \\ &\geq \phi(x)y - \phi(x)y = 0 \end{aligned}$$

and taking the limit we see that $T(x) - S(x) \geq 0$. We can see similarly that $T(x) + S(x) \geq 0$ if $x \in X_+$ so that $S \in \mathcal{L}^r(X, Y)$. As we are assuming that $\mathcal{L}^r(X, Y)$ is a lattice, we can deduce that $T \geq |S|$. For any $\beta \in \mathfrak{b}$ we have

$$|S|(e) = |S|(|e_\beta|) = |S|(e_\beta^+ + e_\beta^-) \geq S(e_\beta^+) + (-S)(e_\beta^-) = S(e_\beta^+ - e_\beta^-) = S(e_\beta)$$

But $S(e_\beta) = \sum_{\alpha \in \mathfrak{b}} \phi_\alpha(e_\beta) y_\beta = y_\beta$, so that $|S|(e)$ is an upper bound for the family of all $(y_\beta)_{\beta \in \mathfrak{b}}$. On the other hand, we have seen that $|S|(e) \leq T(e) = \phi(e)y = y$, where y was any upper bound for $(y_\beta)_{\beta \in \mathfrak{b}}$, so we see that $|S|(e)$ is the supremum of that family, as claimed. Hence Y is Dedekind $\mathfrak{b}$ -complete and hence is Dedekind $<\mathfrak{a}$ -complete. $\square$

The formulation of the preceding result involving $<\mathfrak{a}$ -completeness seems rather artificial. We could hope to replace it by $\mathfrak{b}$ -completeness where $\mathfrak{b}$ is the supremum of density characters of order intervals in X . There is no prospect of proving such a result unconditionally, but there is with a set-theoretic assumption.

Theorem 4.16. *Assume that there is no weakly inaccessible cardinal. Let X be a Banach lattice with an order continuous norm and Y be any Banach lattice. The following are equivalent:*

- (i) *Either X is atomic with an order continuous norm or Y is Dedekind $\mathfrak{a}$ -complete where $\mathfrak{a}$ is the smallest cardinal that is greater than or equal to the density character of every order interval in X .*
- (ii) *$\mathcal{L}^b(X, Y)$ is a Banach lattice under the regular norm in which all lattice operations satisfy the Riesz-Kantorovich formulae.*
- (iii) *$\mathcal{L}^r(X, Y)$ is a vector lattice.*

Proof. The only novel factor here is the proof that (iii) implies (i), given that there are no weakly inaccessible cardinals. If $\mathfrak{b}$ is the smallest cardinal that is greater than the density character of every order interval in X , then certainly $\mathfrak{b} \geq \mathfrak{a}$, as $\mathfrak{a}$ is the smallest cardinal that is greater than or equal to these density characters. By Theorem 4.15 if X is not atomic then Y is Dedekind $<\mathfrak{b}$ -complete. If $\mathfrak{a} < \mathfrak{b}$ then Y is certainly Dedekind $\mathfrak{a}$ -complete. On the other hand, if $\mathfrak{a} = \mathfrak{b}$ then there can be no order interval of density character $\mathfrak{a}$. Certainly $\mathfrak{a}$ will be a limit cardinal and must certainly also be uncountable. If it were regular then it would be weakly inaccessible and therefore can not exist. So we may assume that $\mathfrak{a}$ is singular. Then Theorem 1.15 tells us that Y is Dedekind $\mathfrak{a}$ -complete. $\square$

On the other hand, if there is a weakly inaccessible cardinal, then such a result must fail.

Example 4.17. *Suppose that $\mathfrak{a}$ is a weakly inaccessible cardinal. There exist an AL-space X and AM-space Y such that*

- (i) *$\mathfrak{a}$ is the smallest cardinal such that every order interval in X is $\mathfrak{a}$ -separable.*
- (ii) *$\mathcal{L}^r(X, Y)$ is a vector lattice, but*
- (iii) *X is not atomic and Y is not Dedekind $\mathfrak{a}$ -complete.*

Proof. Let τ be the ordinal such that $\mathfrak{a} = \aleph_\tau$. We take $X = \ell_1(L_1(\mathbf{2}^{\aleph_\sigma}, \gamma^{\aleph_\sigma}))_{\sigma \in \tau}$. Each $L_1(\mathbf{2}^{\aleph_\sigma}, \gamma^{\aleph_\sigma})$ is $\aleph_\sigma$ -homogeneous. It follows easily that $\mathfrak{a}$ is the

smallest cardinal such that every order interval in X is $\mathfrak{a}$ -separable. In fact every element of X lies in the sum of only countably many of these factors, so any order interval lies within such a countable sum. As $\mathfrak{a}$ is uncountable and regular, the supremum ρ of any countable family $\{\sigma_n : n \in \mathbb{N}\}$ will be less than τ . It follows that any order interval has density character at most $\aleph_\rho < \aleph_\tau = \mathfrak{a}$. Thus every order interval in X has density character strictly less than $\mathfrak{a}$. Note that X is certainly not purely atomic.

Take $Y = c(\mathfrak{a}; \mathfrak{a})$ which, by Example 1.13, is Dedekind $<\mathfrak{a}$ -complete but not Dedekind $\mathfrak{a}$ -complete. Theorem 4.15 tells us that $\mathcal{L}^r(X, Y)$ is a vector lattice. $\square$

Open Problem 4.18. It is still an open question exactly when the space $\mathcal{L}^r(X, Y)$ is a vector lattice. The results above suggest that this might be the case if and only if either X is atomic with an order continuous norm or if Y is Dedekind $<\mathfrak{a}$ -complete where $\mathfrak{a}$ is the smallest cardinal that is greater than the density character of every order interval in X . Notice that a consequence of this would be that all lattice operations would be given by the Freudenthal-Kantorovich-Riesz formulae.

There is one further result that, to some extent stands apart from those we have already encountered, but is one that will of interest (albeit disappointingly so) to those interested in ordered Banach algebras. Namely, Andreu, Mazón and Segura de León showed in [2] that, subject to a very mild restriction on the size of a compact Hausdorff space K , $\mathcal{L}^r(C(K))$ is only a lattice when K is Stonean. These notes are probably long enough already without including that result as well. The interested reader is referred to the original paper but is warned that there is a minor gap in the proof there.

It is obvious to ask about stronger order theoretic conditions. These turn out to be easily answered, at least relative to the question of when we have a lattice. The classical case when Y is Dedekind complete actually gives $\mathcal{L}^r(X, Y)$ being Dedekind complete. An identical argument shows that, provided $\mathcal{L}^r(X, Y)$ is a vector lattice then, for any infinite cardinal $\mathfrak{a}$, if Y is Dedekind $\mathfrak{a}$ -complete then so is $\mathcal{L}^r(X, Y)$. On the other hand, by fixing a non-zero positive functional f on X and considering operators $x \mapsto f(x)y$, it is easy to show the converse. Thus:

Theorem 4.19. *If X and Y are Banach lattices such that $\mathcal{L}^r(X, Y)$ is a vector lattice and $\mathfrak{a}$ is any infinite cardinal then $\mathcal{L}^r(X, Y)$ is Dedekind $\mathfrak{a}$ -complete if and only if Y is Dedekind $\mathfrak{a}$ -complete.*

5. WHEN DO THE REGULAR OPERATORS HAVE THE RIESZ SEPARATION PROPERTY?

There is at least one order theoretic condition weaker than that of being a vector lattice that is of interest on spaces of operators. Recall the following definitions:

Definition 5.1. An ordered vector space X has the *Riesz separation property* (RSP) if given $x_1, x_2, z_1, z_2 \in X$ with $x_1, x_2 \leq z_1, z_2$ there is $y \in X$ with $x_1, x_2 \leq y \leq z_1, z_2$. X is said to have the *Riesz decomposition property* (RDP) if given $x, y_1, y_2 \in X$ with $0 \leq x \leq y_1 + y_2$ there are $x_1, x_2 \in X$ such that $x = x_1 + x_2$, $0 \leq x_1 \leq y_1$ and $0 \leq x_2 \leq y_2$.

These two properties are equivalent and are satisfied by any vector lattice.

The proof that we give in Theorem 5.4 that (i) implies (ii) is (an obvious modification of) that given by Danet [4]. That relied on a variant of the *Hahn-Banach-Kantorovich Extension Theorem* for separable domains. This can be actually be extended to the $\mathfrak{a}$ -separable setting, and we do so here.

Recall that if E is a vector space and Y is any vector lattice then a mapping $p : E \rightarrow Y$ is *sublinear*, if for all $x, y \in E$ and for all $\lambda \geq 0$ we have $p(x + y) \leq p(x) + p(y)$ and $p(\lambda x) = \lambda p(x)$. The classical Hahn-Banach-Kantorovich theorem states that if Y is Dedekind complete, E_0 is a linear subspace of E and $p : E \rightarrow Y$ is sublinear then a linear operator $T_0 : E_0 \rightarrow Y$ such that $T_0(x) \leq p(x)$ for all $x \in E_0$ can be extended to a linear operator $T : E \rightarrow Y$ such that $T(x) \leq p(x)$ for all $x \in E$. For some of our results later we need to assume continuity of p , so we need to characterize the continuity of sublinear maps.

Proposition 5.2. *If E is a normed space and Y a normed lattice then a sublinear map $p : E \rightarrow Y$ is continuous if and only if there is a constant K such that $\|p(x)\| \leq K\|x\|$ for all $x \in E$.*

Proof. If there is no such constant then for all $n \in \mathbb{N}$ we can find $x_n \in E \setminus \{0\}$ such that $\|p(x_n)\| \geq n^2\|x_n\|$. Let $z_n = x_n/(n\|x_n\|)$ so that $\|z_n\| = 1/n$ and $z_n \rightarrow 0$. As $p(z_n) = p(x_n/(n\|x_n\|)) \geq (n^2\|x_n\|)/(n\|x_n\|) = n$, we do not have $p(z_n) \rightarrow 0$ and hence p is not continuous.

Conversely, if $\|p(x)\| \leq K\|x\|$ for all $x \in E$ then from $p(x + y) \leq p(x) + p(y)$ we see that $p(x + y) - p(x) \leq p(y)$. Also we have $p(x) \leq p(x + y) + p(-y)$ so that $p(x + y) - p(x) \geq -p(-y)$. It follows from the lattice property of the norm in Y that $\|p(x + y) - p(x)\| \leq K \max\{\|p(y)\|, \|p(-y)\|\} \leq K\|y\|$ showing that p is continuous. $\square$

Theorem 5.3. *Let $\mathfrak{a}$ be an infinite cardinal, X and Y be Banach lattices such that X is $\mathfrak{a}$ -separable and Y has the $\mathfrak{a}$ -interpolation property. Suppose that $p : X \rightarrow Y$ is a continuous sublinear map, X_0 is a vector*

subspace of X , $T_0 : X_0 \rightarrow Y$ is a continuous linear operator such that $T_0(x) \leq p(x)$ for all $x \in X_0$ then T_0 can be extended to a continuous linear operator $T : X \rightarrow Y$ with $T(x) \leq p(x)$ for all $x \in X$.

Proof. Consider the family of all continuous linear extensions of T_0 which are dominated by p on their domains. Order this family by defining $U \prec V$ to mean that V extends U . A routine Zorn's lemma argument gives us a maximal linear extension which is dominated by p . As p is continuous there is, by Proposition 5.2 a constant K such that $\|p(x)\| \leq K\|x\|$ for all $x \in X$. In particular if T_1 is any extension of T_0 dominated by p and x lies in its domain then $T_1(x) \leq p(x)$ and $-T_1(x) = T_1(-x) \leq p(-x)$ so that $-p(-x) \leq T_1(x) \leq p(x)$ and we see that $\|T_1(x)\| \leq K\|x\|$ so that all such extensions are continuous.

It therefore suffices to prove that we can extend the original operator to a larger subspace. Pick $x_1 \in X \setminus X_0$ and let $u, v \in X_0$ be arbitrary. Note that

$$T_0(u) + T_0(v) = T_0(u + v) \leq p(u + v) \leq p(u - x_1) + p(v + x_1)$$

so that

$$T_0(u) - p(u - x_1) \leq p(v + x_1) - T_0(v). \quad (1)$$

As X is $\mathfrak{a}$ -separable there is also a dense subset D of X_0 of cardinality at most $\mathfrak{a}$. The inequality (1) holds in particular for any $u, v \in D$. As Y has the $\mathfrak{a}$ -interpolation property there is $y \in Y$ such that

$$T_0(u) - p(u - x_1) \leq y \leq p(v + x_1) - T_0(v) \quad (2)$$

for all $u, v \in D$. As both p and T_0 are continuous and D is dense, (2) must hold for all $u, v \in X_0$. Define T_1 on the linear span of X_0 and x_1 by defining, if $x_0 \in X_0$ and $\lambda \in \mathbb{R}$, $T_1(x_0 + \lambda x_1) = T_0(x_0) + \lambda y$. If $\lambda = 0$ then we certainly have $T_1(x_0) = T_0(x_0) \leq p(x_0)$. If $\lambda > 0$ then it follows from (2) that $y \leq p(x_0/\lambda + x_1) - T_0(x_0/\lambda)$ so that $\lambda y \leq p(x_0 + \lambda x_1) - T_0(x_0)$ and hence $T_1(x_0 + \lambda x_1) = T_0(x_0) + \lambda y \leq p(x_0 + \lambda x_1)$. Finally if $\lambda < 0$ then (2) this time tells us that $T_0(x_0/(-\lambda)) - p(x_0/(-\lambda) - x_1) \leq y$ so that $T_0(x_0) - p(x_0 + \lambda x_1) \leq -\lambda y$ and hence again we have $T_1(x_0 + \lambda x_1) = T_0(x_0) + \lambda y \leq p(x_0 + \lambda x_1)$. Thus in all three cases T_1 is dominated by p on its domain.

This shows that the domain of a maximal extension must be the whole of X , so this is the desired extension. $\square$

Theorem 5.4. *Let $\mathfrak{a}$ be any infinite cardinal and Y be a Banach lattice with the $\mathfrak{a}$ -interpolation property. If X is an $\mathfrak{a}$ -separable Banach then $\mathcal{L}^r(X, Y)$ has the Riesz Separation Property.*

Proof. We show that $\mathcal{L}^r(X, Y)$ has the Riesz Decomposition Property and hence the Riesz Separation Property. Take three operators $S, T_1, T_2 \in \mathcal{L}^r(X, Y)$ such that $0 \leq S \leq T_1, T_2$. We want to find $S_1, S_2 \in \mathcal{L}^r(X, Y)$ such that $S = S_1 + S_2$ and $0 \leq S_i \leq T_i$ for $i = 1, 2$. Define a map $p : X \times X \rightarrow Y$ by $p(x_1, x_2) = T_1(x_1^+) + T_2(x_2^+)$ for

$x_1, x_2 \in X$. It is routine to show that p is sublinear and continuity of the lattice operations and of each T_i shows that p is continuous for the product topology on $X \times X$.

Let $G = \{(x, x) : x \in X\} \subset X \times X$ and $S_0 : G \rightarrow Y$ by $S_0(x, x) = S(x)$. S_0 is continuous and for all $x \in X$ we have

$$S_0(x, x) = S(x) \leq S(x^+) \leq T_1(x^+) + T_2(x^+) = p(x, x).$$

Use Theorem 5.3 to extend S_0 to $\hat{S} : X \times X \rightarrow Y$ with $\hat{S}(x_1, x_2) \leq p(x_1, x_2)$ for $x_1, x_2 \in X$. Define $S_1, S_2 : X \rightarrow Y$ by $S_1(x) = \hat{S}(x, 0)$ and $S_2(x) = \hat{S}(0, x)$.

If $x \in X_+$ then we have

$$\begin{aligned} S_1(x) &= \hat{S}(x, 0) \leq p(x, 0) = T_1(x) \\ -S_1(x) &= S_1(-x) = \hat{S}(-x, 0) \leq p(-x, 0) = 0 \end{aligned}$$

so that $0 \leq S_1(x) \leq T_1(x)$, i.e. $0 \leq S_1 \leq T_1$. Similarly $0 \leq S_2 \leq T_2$. Finally note that for all $x \in X$ we have

$$S(x) = S_0(x, x) = \hat{S}(x, x) = \hat{S}(x, 0) + \hat{S}(0, x) = S_1(x) + S_2(x)$$

so that $S = S_1 + S_2$. □

Open Problem 5.5. Can the hypothesis that X is $\mathfrak{a}$ -separable in Theorem 5.4 be weakened to requiring that each order interval in X is $\mathfrak{a}$ -separable?

For the case $\mathfrak{a} = \aleph_0$ we have a much more precise result. The main reason the converse will work is because of the following technical result.

Theorem 5.6. *Let K be a compact Hausdorff space and A be an open F_σ -subset of K . There are two sequences in the space $C(K)$, of all continuous real-valued functions on K , (f_n) and (g_n) , with the following properties:*

- (i) $0 \leq f_n(k), g_n(k) \leq 1$ for all $k \in K$.
- (ii) If $m \neq n$ then $f_m \wedge f_n = g_m \wedge g_n = 0$.
- (iii) $f_n \wedge g_n = 0$ for all $n \in \mathbb{N}$.
- (iv) $\bigcup_{n=1}^{\infty} (f_n^{-1}(1) \cup g_n^{-1}(1)) = \bigcup_{n=1}^{\infty} \{k : f_n(k) > 0\} \cup \{k : g_n(k) > 0\} = A$.

Proof. Suppose that $A = \bigcup_{n=1}^{\infty} H_n$ where each H_n is closed. By the Tietze extension theorem, for each $n \in \mathbb{N}$ there is $p_n \in C(K)$ with

- $0 \leq p_n(k) \leq 1$ for all $k \in K$,
- $p_n(k) = 1$ for all $k \in H_n$, and
- $p_n(k) = 0$ for all $k \notin A$.

If we define $p = \sum_{n=1}^{\infty} p_n/2^{-n}$ then this series is uniformly convergent so that $p \in C(K)$ and it is clear that $0 \leq p(k) \leq 1$ for all $k \in K$ and that $f(k) > 0$ if and only if $k \in A$.

Let (α_n) be a strictly decreasing sequence of positive reals converging to zero, with $\alpha_1 = \sup p(K)$. Let $F_n = p^{-1}([\alpha_{2n}, \alpha_{2n-1}])$ for each $n \in \mathbb{N}$, so that each F_n is closed and $F_n \cap F_m = \emptyset$ if $m \neq n$. Let also $U_n = p^{-1}((\frac{\alpha_{2n} + \alpha_{2n+1}}{2}, \frac{\alpha_{2n-1} + \alpha_{2n-2}}{2}))$ (setting $\alpha_0 = \alpha_1 + 1$, for example), so that each U_n is open, $F_n \subset U_n$ and we still have $U_m \cap U_n = \emptyset$ if $m \neq n$. We may use Urysohn's lemma to produce $f_n \in C(K)$ with

- $0 \leq f_n(k) \leq 1$ for all $k \in K$,
- $f_n(k) = 1$ for all $k \in F_n$, and
- $f_n(k) = 0$ for all $k \notin U_n$

(if $F_n = \emptyset$ take $f_n = 0$.) The sequence (f_n) certainly does all that is claimed of it in (i) and (ii) of the statement of the theorem.

Similarly working with the closed sets $G_n = p^{-1}([\alpha_{2n+1}, \alpha_{2n}])$ and the open sets $V_n = p^{-1}((\frac{\alpha_{2n+1} + \alpha_{2n+2}}{2}, \frac{\alpha_{2n} + \alpha_{2n-1}}{2}))$ we may find $g_n \in C(K)$ with

- $0 \leq g_n(k) \leq 1$ for all $k \in K$,
- $g_n(k) = 1$ for all $k \in G_n$, and
- $g_n(k) = 0$ for all $k \notin V_n$,

(again taking $g_n = 0$ if $G_n = \emptyset$.) Again the sequence (g_n) does all that is claimed of it in (i) and (ii).

Claim (iv) in the theorem is because

$$\begin{aligned}
A &\supseteq \bigcup_{n=1}^{\infty} \{k : f_n(k) > 0\} \cup \{k : g_n(k) > 0\} \\
&\supseteq \bigcup_{n=1}^{\infty} (f_n^{-1}(1) \cup g_n^{-1}(1)) \\
&\supseteq \bigcup_{n=1}^{\infty} (F_n \cup G_n) \\
&\supseteq p^{-1}((0, \infty)) \quad (\text{as } \alpha_n \downarrow 0) \\
&= A.
\end{aligned}$$

This only leaves claim (iii) to be verified. Note that $V_n \cap U_m = \emptyset$, and hence $f_m \wedge g_n = 0$, unless $m = n$ or $m = n + 1$. Adding two zero functions at the start of the sequence (g_m) does not alter what we have already established and ensures that f_n will be disjoint from g_n . $\square$

We also need to know which operators on c are positive. In the following proof, e_n is the usual n 'th basic vector and $\mathbf{1}$ the constantly one sequence and we write elements of c as functions on $\mathbb{N}$.

Proposition 5.7. *A bounded linear operator T from c into a Banach lattice Y is positive if and only if (i) $T\mathbf{e}_n \geq 0$ for all $n \in \mathbb{N}$ and (ii) $T(\mathbf{1} - \sum_{n \in F} \mathbf{e}_n) \geq 0$ for all finite sets $F \subset \mathbb{N}$.*

Proof. The proof of “only if” is clear as each $\mathbf{e}_n \geq 0$ and $\mathbf{1} - \sum_{n \in F} \mathbf{e}_n$ is in c and is positive. Suppose that these inequalities are satisfied and that $f \in c_+$. If F is a finite subset of I , put

$$f^F(n) = \begin{cases} f(n) & \text{if } n \in F \\ \ell_f & \text{if } n \notin F \end{cases}$$

(where ℓ_f denotes the limit of the sequence f .) For any $\epsilon > 0$ there is a finite set $F \subset \mathbb{N}$ such that $|f(n) - \ell_f| < \epsilon$ if $n \notin F$ and hence we will have $\|f^F - f\|_\infty \leq \epsilon$. If we show that $T(f^F) \geq 0$ then the continuity of T (and the fact that Banach lattices have closed positive cones) will guarantee that $T(f) \geq 0$.

As $f \geq 0$ we have, for each $n \in \mathbb{N}$, $-\ell_f \leq f(n) - \ell_f$ and hence (by (i)) $-\ell_f T(\mathbf{e}_n) \leq (f(n) - \ell_f)T(\mathbf{e}_n)$. If $F \subset I$ is finite, then note that $f^F = \ell_f \mathbf{1}_I + \sum_{n \in F} (f(n) - \ell_f) \mathbf{e}_n$, so that

$$\begin{aligned} T(f^F) &= T(\ell_f \mathbf{1}) + \sum_{n \in F} (f(n) - \ell_f) T(\mathbf{e}_n) \\ &\geq \ell_f T(\mathbf{1}) - \sum_{n \in F} \ell_f T(\mathbf{e}_n) \\ &= \ell_f T \left(\mathbf{1} - \sum_{n \in F} \mathbf{e}_n \right) \geq 0 \end{aligned}$$

by (ii) and the fact that $\ell_f \geq 0$. □

Theorem 5.8. *The following conditions on a Banach lattice Y are equivalent:*

- (i) Y has the countable interpolation property.
- (ii) For every separable Banach lattice X , $\mathcal{L}^r(X, Y)$ has the RSP.
- (iii) For some infinite compact metric space K , $\mathcal{L}^r(C(K), Y)$ has the RSP.
- (iv) $\mathcal{L}^r(c, Y)$ has the RSP.

Proof. That (i) implies (ii) follows from Theorem 5.4 and (ii) clearly implies (iii).

Assume that (iii) holds and let (k_n) be a convergent sequence in K . Construct pairwise disjoint open sets U_n with $k_n \in U_n$ and choose $h_n \in C(K)$ with, for each $n \in \mathbb{N}$, $0 \leq h_n \leq 1$, $h_n(k_n) = 1$ and h_n being zero on the complement of U_n . If $\mathbf{a} = (a_n) \in c$ write $a_\infty = \lim_{n \rightarrow \infty} a_n$. The series $\sum_{n=1}^\infty (a_n - a_\infty) h_n$ is uniformly convergent in $C(K)$ so we

may define $\Phi \in \mathcal{L}(c, C(K))$ by

$$\Phi(\mathbf{a}) = a_\infty \chi_K + \sum_{n=1}^{\infty} (a_n - a_\infty) h_n.$$

Clearly $\Phi \geq 0$. Define also $\Psi \in \mathcal{L}(C(K), c)_+$ by $\Psi(f) = (f(k_n))$, so that we clearly have $\|\Phi\| = \|\Psi\| = 1$ and $\Psi \circ \Phi$ is the identity on c .

If $\mathcal{L}^r(C(K), X)$ has the RSP then consider $S, T, V, W \in \mathcal{L}^r(c, X)$ with $S, T \leq V, W$. As $\Phi \geq 0$ we see that $S \circ \Phi, T \circ \Phi \leq V \circ \Phi, W \circ \Phi$ so there is $U \in \mathcal{L}^r(C(K), X)$ with $S \circ \Phi, T \circ \Phi \leq U \leq V \circ \Phi, W \circ \Phi$. Using the positivity of Ψ we have $S = (S \circ \Phi) \circ \Psi, T = (T \circ \Phi) \circ \Psi \leq U \circ \Psi \leq V = (V \circ \Phi) \circ \Psi, W = (W \circ \Phi) \circ \Psi$ so that $\mathcal{L}^r(c, X)$ has the RSP.

Let us now suppose that $\mathcal{L}^r(c, Y)$ has the RSP. Clearly, $\mathcal{L}^r(c, I)$ has the RSP for any principal ideal I in Y . In order to prove that X has the Cantor property, it suffices to prove that every principal ideal in E has the Cantor property. Thus, using Kakutani's representation theorem, we may restrict our attention to the case that $X = C(K)$ for some compact Hausdorff space K . In view of a well known theorem due to Seever, [9], we must prove that K is an F-space, i.e. that any pair of disjoint open F_σ subsets of K have disjoint closures.

Let A and B be disjoint open F_σ 's in K . We must prove that they have disjoint closures. By Theorem 5.6 we can find disjoint non-negative sequences $(s_n), (t_n), (u_n)$ and (v_n) in $C(K)$, lying under the constantly one function, with

$$\bigcup_{n=1}^{\infty} (s_n^{-1}(1) \cup t_n^{-1}(1)) = \bigcup_{n=1}^{\infty} \{p : s_n(p) > 0\} \cup \{p : t_n(p) > 0\} = A$$

and

$$\bigcup_{n=1}^{\infty} (u_n^{-1}(1) \cup v_n^{-1}(1)) = \bigcup_{n=1}^{\infty} \{p : u_n(p) > 0\} \cup \{p : v_n(p) > 0\} = B$$

and $s_n \wedge t_n = u_n \wedge v_n = 0$ for all $n \in \mathbb{N}$.

Define operators $S, T : c \rightarrow C(K)$ with $S\mathbf{e}_n = s_n, S\mathbf{1} = 0, T\mathbf{e}_n = t_n$ and $T\mathbf{1} = 0$. The disjointness of the sequences (s_n) and (t_n) guarantees the existence of such operators. For example if $\alpha_n \rightarrow 0$ and $\epsilon > 0$ choose n_0 such that $|\alpha_k| < \epsilon$ for all $k > n_0$. Now if $m > n > n_0$ then

$$\left\| \sum_{k=n}^m \alpha_k s_k \right\|_{\infty} = \left\| \bigvee_{k=n}^m \alpha_k s_k \right\|_{\infty} = \bigvee_{k=n}^m \|\alpha_k s_k\|_{\infty} = \bigvee_{k=n}^m |\alpha_k| \cdot \|s_k\|_{\infty} < \epsilon.$$

Define also $U, V : c \rightarrow C(K)$ with $U\mathbf{e}_n = s_n + t_n + u_n, V\mathbf{e}_n = s_n + t_n + v_n$, and $U\mathbf{1} = V\mathbf{1} = \mathbf{1}_K$. This time the fact that the operators U and V extend to the whole of c is proved by writing them as $S+T$ plus another operator for which the image of the $(\mathbf{e}_n)$ is a disjoint sequence. We may also define an operator $W : c \rightarrow C(K)$ with $W\mathbf{e}_n = s_n + t_n + u_n + v_n$

and $W\mathbf{1} = 4 \times \mathbf{1}_K$. Using Proposition 5.7 we can verify easily that $W \geq S, T, U, V, 0$ from which it follows that $S, T, U, V \in \mathcal{L}^r(c, Y)$.

Clearly $S\mathbf{e}_n, T\mathbf{e}_n \leq U\mathbf{e}_n, V\mathbf{e}_n$ for all $n \in \mathbb{N}$. We also have, for each $n \in \mathbb{N}$,

$$\begin{aligned} (U - S)(\mathbf{1} - \sum_{k=1}^n \mathbf{e}_k) &= U\mathbf{1} - S\mathbf{1} + \sum_{k=1}^n S\mathbf{e}_k - \sum_{k=1}^n U\mathbf{e}_k \\ &= \mathbf{1}_K + 0 + \sum_{k=1}^n s_k - \sum_{k=1}^n (s_k + t_k + v_k) \\ &= \mathbf{1}_K - \sum_{k=1}^n (t_k + v_k) \geq 0 \end{aligned}$$

as all the functions t_k and v_k for $1 \leq k \leq n$ are disjoint, non-negative and lie below $\mathbf{1}_K$. Thus $S \leq U$ and similarly we can show that $S, T \leq U, V$. By hypothesis there is a linear operator $R : c \rightarrow C(X)$ with $S, T \leq R \leq U, V$. Note first that

$$S\mathbf{e}_n, T\mathbf{e}_n \leq R\mathbf{e}_n \leq U\mathbf{e}_n, V\mathbf{e}_n \quad \forall n \in \mathbb{N}$$

so that

$$s_n, t_n \leq R\mathbf{e}_n \leq s_n + t_n + v_n, s_n + t_n + v_n$$

and hence

$$s_n \vee t_n = s_n + t_n \leq R\mathbf{e}_n \leq s_n + t_n + (u_n \wedge v_n) = s_n + t_n$$

showing that $R\mathbf{e}_n = s_n + t_n$. At this stage, let us ask what we know about $R\mathbf{1}$. For each $n \in \mathbb{N}$

$$\begin{aligned} S(\mathbf{1} - \sum_{k=1}^n \mathbf{e}_k), T(\mathbf{1} - \sum_{k=1}^n \mathbf{e}_k) &\leq R(\mathbf{1} - \sum_{k=1}^n \mathbf{e}_k) \\ &= R\mathbf{1} - \sum_{k=1}^n (s_k + t_k) \\ &\leq U(\mathbf{1} - \sum_{k=1}^n \mathbf{e}_k), V(\mathbf{1} - \sum_{k=1}^n \mathbf{e}_k). \end{aligned}$$

That is,

$$-\sum_{k=1}^n s_k, -\sum_{k=1}^n t_k \leq R\mathbf{1} - \sum_{k=1}^n (s_k + t_k) \leq \mathbf{1}_K - \sum_{k=1}^n (s_k + t_k + u_k), \mathbf{1}_K - \sum_{k=1}^n (s_k + t_k + v_k)$$

so that

$$\sum_{k=1}^n s_k, \sum_{k=1}^n t_k \leq R\mathbf{1} \leq \mathbf{1}_K - \sum_{k=1}^n u_k, \mathbf{1}_K - \sum_{k=1}^n v_k.$$

If, for some $n \in \mathbb{N}$ and some $p \in K$ we have $s_n(p) = 1$ or $t_n(p) = 1$ then this shows us that $R\mathbf{1}(p) = 1$ as if, for example, $s_n(p) = 1$ then

$$1 \leq \sum_{k=1}^n s_k(p) \leq R\mathbf{1}(p) \leq \mathbf{1}_K(p) = 1,$$

and similarly $R\mathbf{1}(p) = 0$ if $u_n(p) = 1$ or $v_n(p) = 1$. That is, the function $R\mathbf{1}$, which is an element of $C(K)$, takes the value 1 on A and 0 on B . This certainly suffices to prove that $\overline{A} \cap \overline{B} = \emptyset$ and hence that K is an F-space. $\square$

Open Problem 5.9. Is there a characterisation of Banach lattices with the α -interpolation property similar to the characterisation of those with the countable interpolation problem in Theorem 5.8?

None of these results giving the RSP tell us anything about the norm. There is something that we can say, but only in a very specialised setting. In this proof, $\mathcal{J}$ denotes the natural embedding of $C(K)$ into its bidual.

Theorem 5.10. *Let M be any compact metric space and K a compact F-space, so that $C(K)$ has the countable interpolation property. Every bounded linear operator $T : C(M) \rightarrow C(K)$ is regular with $\|T\|_r = \|T\|$.*

Proof. Fix $y_0 = \|T\|\mathbf{1}_K$ so that $T(C(M)_1) \subset [-y_0, y_0]$. As $C(K)^{**}$ is Dedekind complete, $\mathcal{J} \circ T \in \mathcal{L}(C(M), C(K)^{**})$ has a modulus which satisfies the identity

$$|\mathcal{J} \circ T|(f) = \sup_{C(K)^{**}} \{\mathcal{J} \circ T(g) : |g| \leq f\}$$

for any $f \in C(M)$. In particular, note that $|\mathcal{J} \circ T|(f) \leq y_0$ provided $\|f\| \leq 1$.

We will prove the result by a Zorn's lemma argument on the set $\mathcal{Z}$ of all pairs $\langle E, U \rangle$, where E is a subspace of $C(M)$ containing the constants and $U : E \rightarrow C(K)$ is a linear operator with $U \geq |\mathcal{J} \circ T|_E$ and $U(\chi_M) = y_0$ (this last condition certainly forces $\|U\| = \|U(\chi_M)\| = \|y_0\| = \|T\|$). That this is actually a set (rather than a proper class) is routine to verify by cardinality arguments and that it is non-empty follows by taking E to be the space of constant functions and U to be the unique linear operator on E with $U(\chi_M) = y_0$. Partially order this class by defining $(E_1, U_1) \succ (E_2, U_2)$ to mean that $E_1 \supseteq E_2$ and $U_1|_{E_2} = U_2$. It is routine to verify the conditions for using Zorn's lemma so that we have a maximal element (E_1, U_1) of this set. We claim that $E = C(M)$, so that U_1 actually satisfies the requirements of the theorem.

If $E_1 \neq C(M)$, let $g \in C(M) \setminus E_1$ and let $G = E_1 \oplus \mathbb{R}g$ be the linear span of E_1 and g . We show how to extend U_1 to V on the whole of G with $V \geq |\mathcal{J} \circ T|_G$ and $\|V\| \leq \|y_0\|$, contradicting the alleged maximality of (E_1, U_1) . Consider the sets $A = \{a \in E_1 : a \geq g\}$

and $B = \{b \in E_1 : b \geq -g\}$. As $C(M)$ is separable, both A and B are separable. Let A' and B' be countable dense subsets of A and B respectively. If $a \in A'$, let C_a be a countable dense subset of the set $\{\phi \in C(M) : |\phi| \leq a - g\}$. Similarly, for $b \in B'$, let D_b be a countable dense subset of the order interval $\{\psi \in C(M) : |\psi| \leq b + g\}$.

If $a \in A'$, $b \in B'$, $\phi \in C_a$ and $\psi \in D_b$ then $a - g \geq 0$ and $b + g \geq 0$ so that certainly $a + b \geq 0$. Hence

$$\begin{aligned} U(a) + U(b) &= U(a + b) \\ &\geq |\mathcal{J} \circ T|(a + b) \\ &= |\mathcal{J} \circ T|(a - g + b + g) \\ &= |\mathcal{J} \circ T|(a - g) + |\mathcal{J} \circ T|(b + g) \\ &\geq |\mathcal{J} \circ T|(|\phi|) + |\mathcal{J} \circ T|(|\psi|) \\ &\geq T(\phi) + Tz(\psi) \end{aligned}$$

so that

$$U(a) - T(\phi) \geq T(\psi) - U(b).$$

This holds simultaneously for any choice of $a \in A'$, $\phi \in C_a$, $b \in B'$ and $\psi \in D_b$. Since there are only countably many such choices, we may use the countable interpolation property in $C(K)$ to find $y \in C(K)$ with

$$U(a) - T(\phi) \geq y \geq T(\psi) - U(b)$$

for all $a \in A$, $\psi \in C_a$, $b \in B$ and $\psi \in D_b$.

Now let us define V on G by $V(f + \alpha g) = U(f) + \alpha y$ for all $f \in E_1$. If $a \in A'$ then, for each $\phi \in C_a$, we have

$$V(a - g) = U(a) - y \geq T(\phi)$$

and by continuity this persists for all ϕ with $0 \leq |\phi| \leq a - g$. Hence, regarded as operators into $C(K)^{**}$, we have

$$V(a - g) \geq \sup_{C(K)^{**}} \{T(\phi) : |\phi| \leq a - g\} = |\mathcal{J} \circ T|(a - g)$$

for all $a \in A'$. Again use the density of A' in A , together with the continuity of U , to see that we actually have

$$V(a - g) = U(a) - y \geq |\mathcal{J} \circ T|(a - g)$$

whenever $a - g \geq 0$. Similarly, if $a + g \geq 0$ then we also have

$$V(a + g) \geq |\mathcal{J} \circ T|(a + g).$$

The linearity of V and of $|T|$ now make it clear that if $f + \alpha g \geq 0$ then $V(f + \alpha g) \geq |T|(f + \alpha g)$ whether $\alpha > 0$, $\alpha < 0$ or $\alpha = 0$ (in which case it is clear). It follows that $V \geq |\mathcal{J} \circ T|_G$, so that, in particular, $V \geq 0$. That means that $\|V\| = \|V\chi_M\| = \|y_0\|$ so that $\langle G, V \rangle \in \mathcal{Z}$ and we have contradicted the alleged maximality of $\langle E_1, U_1 \rangle$.

Thus U_1 is defined on all of $C(M)$ and we have $U \geq |\mathcal{J} \circ T|$ (so that $U \geq \pm T$) and $\|U\| = \|T\|$ showing that $\|T\|_r = \|T\|$. $\square$

Results of Ando [1] and Davies [5] allow us to deduce that in this case the dual, $\mathcal{L}(C(M), C(K))^*$ is a Banach lattice under the dual norm and order.

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