An approximate formula for the relaxation time of a single domain ferromagnetic particle with uniaxial anisotropy and collinear field

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Recently there have been renewed efforts to solve the Fokker–Planck equation resulting from Brown's model [Phys. Rev. **130**, 1677 (1963)] of single domain ferromagnetic particle relaxation for the case of uniaxial anisotropy and zero external field. In particular, the usefulness of a simple analytic formula for the relaxation time based on the reciprocal of the lowest non-zero eigenvalue resulting from this model has been stressed. Here we suggest an improved analytic formula which we extend to the case of applied collinear field and compare it with the exact numerical solution and previous analytic expressions. The results are presented in terms of the pre-factor dependence which is commonly taken as constant. The formula exhibits good agreement with the exact results throughout the full range of anisotropy and reduced external field values.

Brown's model for the relaxation of the magnetization of a single domain ferromagnetic particle is based on the well-known phenomenological Landau–Lifshitz–Gilbert equation.[1] This equation describes the damped precession of the magnetization in a magnetic field. Brown augmented the field in this equation by a random term in order to include the thermal forces involved. He averaged this equation to obtain the Fokker–Planck equation of the process. Solutions of this equation are usually attempted by expanding the probability density of orientations of the magnetization as an infinite series of eigenfunctions and eigenvalues.[2–4] The physical quantity of most interest available from this model is the relaxation time. Provided one non-zero eigenvalue is significantly smaller than the rest, the relaxation time can be taken as proportional to the reciprocal of this lowest non-zero eigenvalue, except in the very early stages of approach to equilibrium. This is true for the cases considered here. The relaxation time then is given by

$$\tau=\frac{v}{kT\eta\lambda_1}\left(\frac{1}{\gamma^2}+\eta^2M_s^2\right), \tag{1}$$

where γ is the gyromagnetic ratio, M_s is the saturation magnetization, v is the particle volume, η is the damping constant from the Landau–Lifshitz–Gilbert equation, kT is thermal energy, and λ_1 is the lowest non-zero eigenvalue.

In the case of uniaxial anisotropy, the magnetization has two minima separated by an energy barrier. The anisotropy parameter is the ratio of anisotropy energy to thermal energy, given by $\sigma=Kv/kT$ where K is the bulk anisotropy constant. The dependence of the eigenvalue λ_1 on σ can be computed numerically,[2–4] but a simple approximate analytic formula which corresponds well to the exact numerical solution is useful, as often the eigenvalue computation is only one of many steps in a larger calculation. Brown rigorously derived simple analytic formulae for this eigenvalue. These are

$$\lambda_1=\frac{4}{\sqrt{\pi}}\sigma^{3/2}e^{-\sigma}, \tag{2}$$

which is the asymptote for $\sigma\gg 1$ and

$$\lambda_1=2-\frac{4\sigma}{5}+\frac{96\sigma^2}{875}, \tag{3}$$

which is an expansion for $\sigma\ll 1$. Both these were later refined.[2,5] However the use of two formulae in different ranges is awkward. A single approximate

$$\lambda_1=2(1+\sigma/4)^{5/2}e^{-\sigma} \tag{4}$$

has been suggested by Bessais *et al.*[6] Expressing concern that for $\sigma\gg 1$ this has a different asymptote to the rigorous derivation of Brown, Aharoni[7] has proposed

$$\lambda_1=2\left(\frac{2+9\sigma/5+(4/\pi)^{1/3}\sigma^2}{2+\sigma}\right)^{3/2}e^{-\sigma}, \tag{5}$$

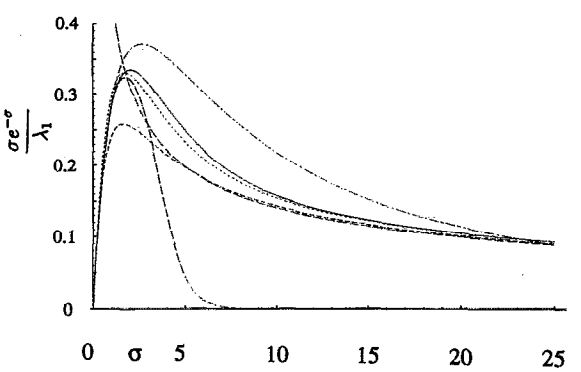

FIG. 1. Normalized pre-factor $\sigma e^{-\sigma}/\lambda_1$ vs σ. The solid line is the exact solution, large dashed line is for Eqs. (2) and (3), mixed dashed line is for Eq. (4), medium dashed line is for Eq. (5), and small dashed line is for Eq. (6).

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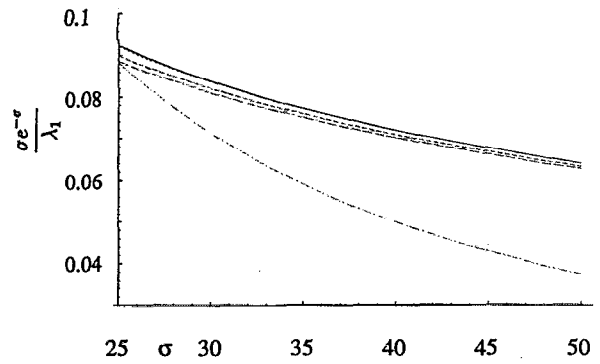

FIG. 2. Same as Fig. 1. Equation (6) lies *very* close to the exact solution, whilst Eq. (4) is not asymptotically correct.

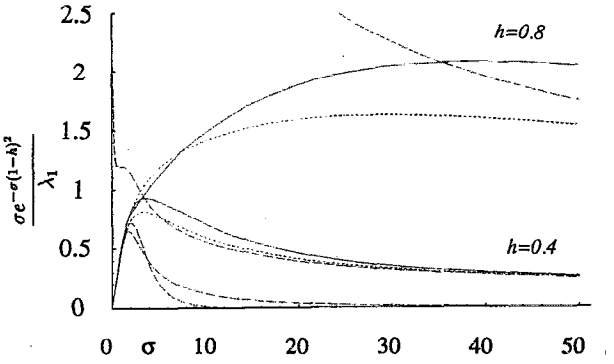

FIG. 4. Same as Fig. 3, with $h=0.4$ and $h=0.8$.

which has the correct asymptote for $\sigma \gg 1$, as well as having agreement with the expansion when $\sigma \ll 1$. In the light of these efforts and as a consequence of applying a variational principle to the problem as outlined originally by Brown, we have recently observed improved agreement with the exact solution from

$$\lambda_1 = \left(\frac{2\sigma\sqrt{\sigma/\pi}}{(\sigma+1)} + 2^{-\sigma} \right) \frac{2\sigma}{e^{\sigma}-1} . \tag{6}$$

These different formulae are compared in Figs. 1 and 2 with the exact solution in the following manner. Noting the definition of σ we rewrite Eq. (1) in the Arrhenius form

$$\tau = \tau_0 e^{\sigma}. \tag{7}$$

Thus the dependence of the pre-factor τ_0 on σ is

$$\tau_0 = \left(\frac{1}{\gamma^2} + \eta^2 M_s^2 \right) \frac{\sigma e^{-\sigma}}{K \eta \lambda_1}. \tag{8}$$

In Figs. 1 and 2 we present $\sigma e^{-\sigma}/\lambda_1$ as a function of σ for the various approximations and the exact solution. The advantage of presenting the results in terms of the pre-factor dependence is that it allows an easy comparison with the commonly used *constant* pre-factor[8] attributed to Néel.[9]

The form of the present approximate allows it to be modified to include the case of a collinear field, that is, an external magnetic field applied along the easy axis of uniaxial anisotropy. For this case Brown presented two rigorous formulae,

$$\lambda_1 = \frac{2}{\sqrt{\pi}} \sigma^{3/2} (1-h^2) \{ (1-h) \exp[-\sigma(1-h)^2] + (1+h) \exp[-\sigma(1+h)^2] \} \tag{9}$$

for $\sigma(1-h)^2 \gg 1$, and

$$\lambda_1 = 2 - \frac{4\sigma}{5} + \frac{96\sigma^2}{875} \left(1 + \frac{175h^2}{24} \right) \tag{10}$$

for $\sigma \ll 1$, where h is the reduced field strength parameter for an external field strength H given by $h = HM_s/2K$. These reduce to Eqs. (2) and (3) for $h=0$. Our modified single suggestion in this case is

$$\lambda_1 = \sigma(1-h^2) \left(\frac{2\sigma\sqrt{\sigma/\pi}}{(\sigma+1)} + 2^{-\sigma} \right) \left(\frac{1-h}{\exp[\sigma(1-h)^2]-1} + \frac{1+h}{\exp[\sigma(1+h)^2]-1} \right), \tag{11}$$

which reduces to Eq. (6) for $h=0$. Here it is advantageous to present the results in terms of the Arrhenius form

$$\tau = \tau_0 \exp[\sigma(1-h)^2]. \tag{12}$$

In Figs. 3–6, we present $\sigma \exp[-\sigma(1-h)^2]/\lambda_1$ for these

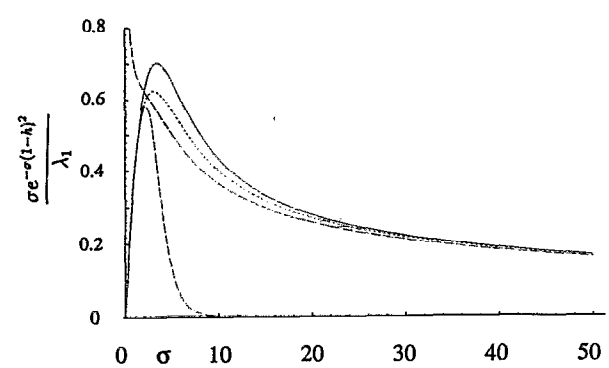

FIG. 3. Normalized pre-factor $\sigma \exp[-\sigma(1-h)^2]/\lambda_1$ vs σ. The solid line is the exact solution, large dashed line is for Eqs. (9) and (10), and small dashed line is for Eq. (11), $h=0.2$.

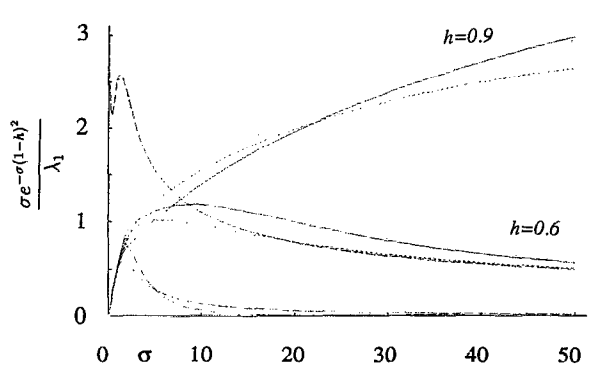

FIG. 5. Same as Fig. 3, with $h=0.6$ and $h=0.9$. Brown's Eq. (9)>10 in this range.

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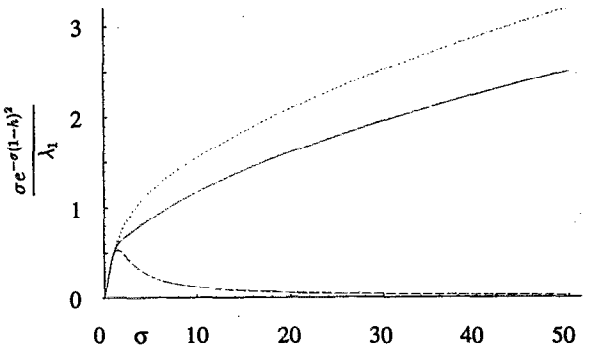

FIG. 6. Same as Fig. 3, with $h=0.99$. Brown's Eq. (9) is larger in this range by 2–3 orders of magnitude.

approximates and the exact solution for various values of the reduced field h. Note, due to the $\sigma/(e^{\sigma}-1)$ term, l'Hôpital's rule needs to be observed for $\sigma\to 0$, so one should avoid $\sigma=0$ in the computation of Eq. (6) and avoid it similarly for Eq. (11).

The suggested formula gives improved agreement in the case of $h=0$ and good agreement with the exact results throughout the full range of anisotropy and reduced applied field. In particular, we note its validity for h close to 1. Significantly, we note from the figures that the exact prefactor for Eq. (12) is not constant, even for $\sigma>25$, but increases by 1–2 orders of magnitude as h increases. The case of collinear applied field is a special one and quite difficult to achieve for an ensemble of particles.[10] However where a high collinearity can be achieved, the new suggestion should be useful.

We note in closing that the problem under consideration is analogous to that of Martin, Meier, and Saupe[11,12] for the relaxation of nematic liquid crystals possessing uniaxial crystalline anisotropy.

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