

Operator theoretic characterisations of completeness in Riesz spaces

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1. Riesz spaces and regular operators.

In this first section, we present just enough of the basic definitions of the theory of Riesz spaces and of operators on them for the purposes of this work. This will merely scratch the surface of the extensive literature on the topic. The interested reader who wishes to find out more on this topic is referred to the books listed in the bibliography at the end of this set of notes.

A relation on a non-empty set S , $\leq$, is an *order* if the following three conditions hold, where x, y and z are allowed to range over the whole of S .

- (a) $x \leq x$.
- (b) $x \leq y$ and $y \leq x$ together imply that $x = y$.
- (c) $x \leq y$ and $y \leq z$ together imply that $x \leq z$.

A subset A of an ordered set S is *bounded above* if there is $s \in S$ such that $a \leq s$ for all $a \in A$. In this case we say that s is an *upper bound* for A . *Lower bounds* and the concept of being *bounded below* are defined by reversing the inequalities. A set is *order bounded* if it is both bounded above and bounded below. An *order interval* in S is a set of the form

$$[x, y] = \{s \in S : x \leq s \leq y\}$$

for some $x, y \in S$. Order bounded sets are precisely those contained in some order interval.

An ordered set $(S, \leq)$ is a *lattice* if any two elements $x, y \in S$ have both a *greatest lower bound* or *supremum*, denoted by $x \wedge y$, and a *least upper bound* or *infimum*, denoted by $x \vee y$. Thus $x \vee y \geq x, y$ and if $z \geq x, y$ then $z \geq x \vee y$. If A is a non-empty set in S then it may or may not have upper and lower bounds. Even if these exist there may or may not be a least upper bound and a greatest lower bound. If these do exist we denote them by $\bigvee A$ and $\bigwedge A$ respectively. If, for example, $\bigvee A$ exists then $\bigvee A \geq a$ for all $a \in A$ whilst if $b \geq a$ for all $a \in A$ then $b \geq \bigvee A$.

A *Riesz space* or *vector lattice* E is a real vector space, together with an order under which it is a lattice, in which the two structures are connected by the two conditions

- (a) If $x, y, z \in E$ and $x \leq y$ then $x + z \leq y + z$,
- (b) If $x, y \in E, \alpha \in \mathbb{R}, x \leq y$ and $\alpha \geq 0$ then $\alpha x \leq \alpha y$.

The *positive cone* of a Riesz space E is the set $E_+ = \{x \in E : x \geq 0\}$. This has the properties of being closed under addition and multiplication by non-negative reals. It completely determines the order structure of E in the sense that $x \geq y \Leftrightarrow x - y \in E_+$. If $x \in E$, its *positive part* is $x^+ = x \vee 0$, its *negative part* is $x^- = (-x) \vee 0$ (which is positive!) and its *modulus* is $|x| = x \vee (-x)$. It is not too difficult to see that $x = x^+ - x^-$ and that $|x| = x^+ + x^-$. Two elements, x and y , of a Riesz space are *disjoint* if $|x| \wedge |y| = 0$.

There is one other condition which we will impose on Riesz spaces which we plan to study. A Riesz space E is *Archimedean* if whenever $x, y \in E$ with $nx \leq y$ for all $n \in \mathbb{N}$ we have $x \leq 0$. This technical condition is not part of the definition of a Riesz space, but we will nevertheless assume that all Riesz spaces under consideration satisfy this condition without subsequently mentioning it explicitly.

A *Banach lattice* E is a Riesz space which is also a Banach space and in which the two structures are connected by the implication $|x| \leq |y| \Rightarrow \|x\| \leq \|y\|$ for all $x, y \in E$.

Before proceeding further let us give an example of a Riesz space (and a very important one at that).

Example 1.1. Let K be a compact Hausdorff space and let $C(K)$ denote the real vector space of all continuous real-valued functions on K . Order $C(K)$ by defining $f \geq g$ to mean that $f(k) \geq g(k)$ for all $k \in K$. It is routine to verify that this makes $C(K)$ into a Riesz space and that the lattice operations are given by the formulae

$$(f \vee g)(k) = \max\{f(k), g(k)\}$$

and $(f \wedge g)(k) = \min\{f(k), g(k)\}.$

In particular it follows that $|f|(k) = (f \vee (-f))(k) = \max\{f(k), -f(k)\} = |f(k)|$. When equipped with the supremum norm,

$$\|f\|_\infty = \sup\{|f(k)| : k \in K\}$$

it is well-known that $C(K)$ is a Banach space. In order to see that we actually have a Banach lattice, we need only note that if $|f| \leq |g|$ then for each $k \in K$ we have $|f(k)| \leq |g(k)| \leq \|g\|_\infty$, and taking a supremum as k varies over the whole of K we see that $\|f\|_\infty \leq \|g\|_\infty$.

This example is a very important one for various reasons. It is also one of the oldest to be studied systematically and an abstract characterisation of such spaces has been known since the early 1940's. An *order unit* for a Riesz space E is an element $e \in E$ such that for each $x \in E$ there is $0 \leq \lambda \in \mathbb{R}$ such that $|x| \leq \lambda e$. An order unit e induces a norm on E by defining

$$\|x\|_e = \inf\{\lambda \in \mathbb{R} : |x| \leq \lambda e\}.$$

On $C(K)$ the constantly one function is an order unit (remember that on compact Hausdorff spaces continuous functions must be bounded) and the norm that it induces is simply the usual supremum norm. Conversely we have:

Theorem 1.2. *If E is a Riesz space with an order unit e and E is complete under the norm $\|\cdot\|_e$ then there is a compact Hausdorff space K and a bijective linear operator $J : E \rightarrow C(K)$ such that $\|Jx\|_\infty = \|x\|_e$ for all $x \in E$ and $Jx \geq Jy \Leftrightarrow x \geq y$ for all $x, y \in E$.*

A proof of this may be found in Theorem 2.1.3 of [7]. This result means that we can (essentially) identify such spaces with a $C(K)$ -space as far as the norm and order are concerned and then we have all the classical tools available to study them. It is always possible to find a lot of subspaces of a Riesz space with an order unit—simply choose your order unit e and study $E_e = \{x : \exists \lambda \in \mathbb{R} : |x| \leq \lambda e\}$, the ideal generated by e . The problem is ensuring completeness of E_e under the norm $\|\cdot\|_e$.

This may be an appropriate time to introduce a number of definitions of completeness properties that we will be referring to repeatedly in what is to come. A Riesz space is *Dedekind complete* if every non-empty subset which is bounded above has a supremum. The corresponding statement for lower bounds follows automatically. E is *Dedekind σ -complete* if every countable subset with an upper bound has a supremum. E has the *countable interpolation property* if whenever (x_n) and (z_n) are sequences in E , with $x_n \uparrow$ and $z_n \downarrow$, such that $x_m \leq z_n$ for all $m, n \in \mathbb{N}$ there exists $y \in E$ with $x_n \leq y \leq z_n$ for all $n \in \mathbb{N}$. Notice that this notion makes sense for any ordered set, not just for Riesz spaces. A rather stronger notion is that of the *strong countable interpolation property*, the definition of which is the same as that of the countable interpolation property except for omitting the hypotheses that the two sequences are monotone. For Riesz spaces the two notions are easily seen to be equivalent, but in general they are not. Finally we say that a Riesz space E is *uniformly complete* if for every $e \in E_+$ the normed space $(E_e, \|\cdot\|_e)$ is norm complete.

Clearly Dedekind complete Riesz spaces are Dedekind σ -complete. Similarly Dedekind σ -complete Riesz spaces have the countable interpolation property (just take $y = \bigvee \{x_n : n \in \mathbb{N}\}$).

Proposition 1.3. *Every Riesz space with the countable interpolation property and every Banach lattice is uniformly complete.*

A proof of this is included in Proposition 1.1.8 of [7]. None of the implications that we have established can be reversed. There are examples of Dedekind σ -complete Riesz spaces which are not Dedekind complete, of Riesz spaces with the countable interpolation property which are not Dedekind σ -complete and of uniformly complete Riesz spaces which do not have the countable interpolation property or which cannot be made into Banach lattices. We have no need of such examples for what follows so we refer the interested reader to the literature (especially [6, Ch4, §5]) for examples.

In many Riesz spaces and in all Banach lattices the subspaces E_e may thus be identified with certain spaces $C(K)$. Many properties are held by Riesz spaces if and only if they are held by each subspace E_e . For example we have:

Proposition 1.4. *A Riesz space E is Dedekind complete if and only if every set E_e is Dedekind complete whenever $0 \leq e \in E$.*

Proof. Suppose that E is Dedekind complete, that $0 \leq e \in E$ and that $A \subset E_e$ is bounded above by $m \in E_e$. We can form $s = \bigvee A$ in E and it will be in E_e . To see this, choose any $a \in A$ and then pick $\lambda, \mu \in \mathbb{R}$ with $|a| \leq \lambda e$ and $|m| \leq \mu e$. We have $a \leq s \leq m$ and hence $|s| \leq |a| \vee |m| \leq (\lambda e) \vee (\mu e) = \max\{\lambda, \mu\}e$. Clearly s

is an upper bound for A in the subspace E_e . If t is any other upper bound for A in E_e then it will also be an upper bound for A in E . But s is the least of all such upper bounds, so $s \leq t$. This shows that s is also the supremum of A when regarded as a subset of E_e .

Now suppose that each E_e is Dedekind complete and that $A \subset E$ is bounded above by m . Pick $a_0 \in A$ and let $A' = \{a \vee a_0 : a \in A\}$. Note that $b \in E$ is an upper bound for A if and only if it is an upper bound for A' , so we may consider only that case. Every element of A' thus lies between a_0 and m and it follows that if $a \in A'$ then $|a| \leq |a_0| \vee |m|$. I.e. $A' \subseteq E_{|a_0| \vee |m|}$ and A' is bounded above in $E_{|a_0| \vee |m|}$ by m . We may thus find a supremum for A' in $E_{|a_0| \vee |m|}$, (say) s . Clearly s is an upper bound for A' in E . If t is any other upper bound for A' in E then so is $t \wedge m$, which lies in $E_{|a_0| \vee |m|}$ as we certainly have $t \geq a_0$ and hence $m \geq t \wedge m \geq a_0$. Thus $t \geq t \wedge m \geq s$, confirming that s is the supremum of A' (and hence of A) in the whole of E . ■

Similar statements hold for Dedekind σ -completeness and for the countable interpolation property. It is thus of considerable interest to know when $C(K)$ -spaces have these various completeness properties.

Before stating the results we need some definitions. A subset of a topological space is an F_σ if it is a countable union of closed sets. A compact Hausdorff space is *Stonean* if the closure of every open subset is open or, equivalently, any two disjoint open sets have disjoint closures. A compact Hausdorff space is *quasi-Stonean* if the closure of every open F_σ is open, whilst it is an F -space if two disjoint open F_σ sets have disjoint closures. The last two definitions are **not** equivalent. It is not easy to give non-trivial examples of any of these kind of spaces (the simplest infinite Stonean space is the Stone-Ćech compactification of $\mathbb{N}$). It is not too difficult to see that compact metric spaces satisfying any one of these conditions must be finite.

Theorem 1.5. *The following three conditions on a compact Hausdorff space K are equivalent:*

- (i) K is Stonean.
- (ii) $C(K)$ is Dedekind complete.
- (iii) Any non-empty set of disjoint elements which is bounded above has a supremum.

Very few texts include a proof of this as the proof is rather simpler than (and very similar to) that of the next result.

Theorem 1.6. *The following three conditions on a compact Hausdorff space K are equivalent:*

- (i) K is quasi-Stonean.
- (ii) $C(K)$ is Dedekind σ -complete.
- (iii) Any countable non-empty set of disjoint elements which is bounded above has a supremum.

This is Proposition 2.1.5 of [7]. As an immediate corollary of the last two results and Theorem 1.3 we obtain the following result due to Veksler and Gejler.

Corollary 1.7. *Let E be a uniformly complete Riesz space.*

- (a) E is Dedekind complete if and only if every disjoint set which is bounded above has a supremum.
- (b) E is Dedekind σ -complete if and only if every disjoint countable set which is bounded above has a supremum.

As a final completeness characterisation, we have:

Theorem 1.8. *The following two conditions on a compact Hausdorff space K are equivalent:*

- (i) K is an F -space.
- (ii) $C(K)$ has the countable interpolation property.

Although very similar in style to the proof of Theorem 1.6, this result has not made its way into many of the standard texts. The interested reader is referred to the original paper of Seever, [9], for the details.

If E and F are Riesz spaces, then a linear operator $T : E \rightarrow F$ is *positive* if $x \geq 0 \Rightarrow Tx \geq 0$ for all $x \in E$. The linear span of the positive operators forms the linear space of *regular* operators, which we denote by $L^r(E, F)$. It is only in some rather special cases that $L^r(E, F)$ is a Riesz space. Some of the results in this course are concerned precisely with conditions under which this does occur. There is, in general, no simple description of regular operators. There is, however, a simply described property that they possess. A linear operator $T : E \rightarrow F$ is *order bounded* if $T(A)$ is order bounded in F whenever A is an order bounded subset of E . An equivalent formulation is that for every $x \in E_+$ there is $y \in F_+$ with $T([-x, x]) \subseteq [-y, y]$.

Proposition 1.9. *If E and F are Riesz spaces and $T : E \rightarrow F$ is a regular linear operator then T is order bounded.*

Proof. Suppose that $T = U - V$ where $U, V : E \rightarrow F$ are positive linear operators. If $0 \leq x \in E$ and $|a| \leq x$ then we also have $a^\pm \leq x$. It follows that

$$Ta = Ta^+ - Ta^- \leq Ua^+ + Va^- \leq (U + V)x.$$

The same argument applies to $-a$, so we have $|Ta| \leq (U + V)x$. I.e. we have shown that $T([-x, x]) \subseteq [-(U + V)x, (U + V)x]$. ■

Finally, in this introductory section, let us record:

Proposition 1.10. *If E and F are Banach lattices and $T : E \rightarrow F$ is a regular linear operator then T is norm bounded.*

This is proved in Proposition 1.3.6 of [7].

2. Characterisations of Dedekind completeness.

Our first result is possibly the easiest of all our results to state and is also (historically) the first result along these lines, being due to Abramovich and Gejler in 1982. Their result was essentially a converse of a classical result of Kantorovich *et al.* which dates back to the 1930's.

The proof of this result requires the use of a class of Riesz spaces which will be used several times during this course. We take the opportunity to make a formal definition now.

Definition 2.1. Given any set I , we will denote by $\ell_0^\infty(I)$ the space of all real-valued functions on I which are constant except on a finite set. When given the usual linear operations and the pointwise partial order this is always a Riesz space. We will denote the constantly one function in $\ell_0^\infty(I)$ by $\mathbf{1}$ and use $\mathbf{e}_i$ to denote the characteristic function of $\{i\}$. Note that the set $\{\mathbf{e}_i : i \in I\} \cup \{\mathbf{1}\}$ is a Hamel basis for $\ell_0^\infty(I)$, so that we may specify a linear operator on $\ell_0^\infty(I)$ by specifying its value at each element of that set.

Operators on spaces like this will occur several times in the course of our discussions, so let us record now when such operators are positive.

Proposition 2.2. *If F is a Riesz space, I is any non-empty set and $T : \ell_0^\infty(I) \rightarrow F$ then T is positive if and only if*

- (a) $T\mathbf{e}_i \geq 0$ for all $i \in I$
- (b) $T\mathbf{1} \geq \sum_{i \in F} T\mathbf{e}_i$ for all finite subsets F of I .

Proof. The fact that T being positive implies (a) and (b) is simply because $\mathbf{e}_i$ and each of the terms $\mathbf{1} - \sum_{i \in F} \mathbf{e}_i$ is positive, so their images must be. Now suppose that (a) and (b) hold and that $x = \alpha\mathbf{1} + \sum_{i \in F} \beta_i \mathbf{e}_i \geq 0$, where F is some finite subset of I . As x takes the value α at any $i \in I \setminus F$, we see that

$\alpha \geq 0$. Similarly, if we look at the value of x at $i \in F$ we see that $\alpha + \beta_i \geq 0$. I.e. $\beta_i \geq -\alpha$ for all $i \in F$. Now we see that

$$\begin{aligned}
Tx &= \alpha T\mathbf{1} + \sum_{i \in F} \beta_i T\mathbf{e}_i \\
&\geq \alpha T\mathbf{1} - \sum_{i \in F} \alpha T\mathbf{e}_i \\
&= \alpha \left(T\mathbf{1} - \sum_{i \in F} T\mathbf{e}_i \right) \\
&\geq 0
\end{aligned}$$

where the first inequality is because $\beta_i \geq -\alpha$ and (a), whilst the second is because $\alpha \geq 0$ and (b). ■

A related result will be needed later, so we get it on record now.

Corollary 2.3. *If F is a Banach lattice, I is any non-empty set with the discrete topology, $\gamma(I)$ its one-point compactification and $T : C(\gamma(I)) \rightarrow F$ is a norm bounded linear operator then T is positive if and only if*

(a) $T\mathbf{e}_i \geq 0$ for all $i \in I$

(b) $T\mathbf{1} \geq \sum_{i \in F} T\mathbf{e}_i$ for all finite subsets F of I .

Proof. An obvious consequence of Proposition 2.2 once we note that $\ell_0^\infty(I)$ is norm-dense in $C(\gamma(I))$ and use the norm boundedness of T . ■

Theorem 2.4. *If F is a Riesz space then the following are equivalent.*

- (i) F is Dedekind complete.
- (ii) For all Riesz spaces E , $L^r(E, F)$ is a Dedekind complete Riesz space.
- (iii) For all Riesz spaces E , $L^r(E, F)$ is a Riesz space.

Proof. It is Kantorovich's classical theorem that (i) $\Rightarrow$ (ii). Proofs of this may be found in many text books, including Theorem 1.3.2 of [7]. Clearly (ii) $\Rightarrow$ (iii), so we need only prove that (iii) $\Rightarrow$ (i).

Assuming that (iii) holds, we show first that F is Dedekind σ -complete. In order to prove this it suffices to consider an increasing sequence of positive elements of F , which is bounded above, and prove that it has a supremum. Let $0 \leq a_n \uparrow$ and let $a_n \leq b$ for all $n \in \mathbb{N}$. Let us define $T, U \in L^r(\ell_0^\infty(\mathbb{N}), F)$ as follows:

$$\begin{aligned} T\mathbf{e}_1 &= U\mathbf{e}_1 = a_1 \\ T\mathbf{e}_n &= U\mathbf{e}_n = a_n - a_{n-1} \quad (n > 1) \\ T\mathbf{1} &= 0 \\ U\mathbf{1} &= b \end{aligned}$$

I claim that U is a positive linear operator. This follows from Proposition 2.2 as soon as we observe that each $U\mathbf{e}_n \geq 0$ and that for any finite set $F \subseteq \mathbb{N}$ there is $N \in \mathbb{N}$ with $F \subseteq \{1, 2, \dots, N\}$ and that we then have

$$\begin{aligned} \sum_{n \in F} U\mathbf{e}_n &\leq \sum_{n=1}^N U\mathbf{e}_n \\ &= a_1 + \sum_{n=2}^N (a_n - a_{n-1}) \\ &= a_N \\ &\leq b = U\mathbf{1}. \end{aligned}$$

It is even easier to see that $U - T$ is positive. It follows that T is a regular operator, since $T = U - (U - T)$ and both U and $U - T$ are positive. By hypothesis, the space $L^r(\ell_0^\infty(\mathbb{N}), F)$ is a Riesz space, so T^+ exists in $L^r(\ell_0^\infty(\mathbb{N}), F)$. What is $T^+\mathbf{1}$? For each $N \in \mathbb{N}$ we have $\mathbf{1} \geq \sum_{n=1}^N \mathbf{e}_n \geq 0$, so that

$$\begin{aligned} T^+\mathbf{1} &\geq T^+ \left(\sum_{n=1}^N \mathbf{e}_n \right) \\ &\geq T \left(\sum_{n=1}^N \mathbf{e}_n \right) \\ &= a_N \end{aligned}$$

so that $T^+\mathbf{1}$ is an upper bound for $\{a_n : n \in \mathbb{N}\}$. On the other hand $T^+ \leq U$, so that $T^+\mathbf{1} \leq U\mathbf{1} = b$. But in the arguments above we could have taken b to be any upper bound for $\{a_n : n \in \mathbb{N}\}$, so that $T^+\mathbf{1}$ is the least of all possible upper bounds for this set and thus $\{a_n : n \in \mathbb{N}\}$ has a supremum, namely $T^+\mathbf{1}$.

Now that we know that F is Dedekind σ -complete, we may make use of the Corollary 1.7 in our attempt to prove that F is Dedekind complete. This tells us that we need only prove that every disjoint set $\{y_i : i \in I\}$ of positive elements of F , which is bounded above by y , has a supremum. Again we may define two linear operators, this time from $\ell_0^\infty(I)$ into F , as follows:

$$\begin{aligned} T\mathbf{e}_i &= U\mathbf{e}_i = y_i \\ T\mathbf{1} &= 0 \\ U\mathbf{1} &= y. \end{aligned}$$

Again it is routine to prove that $U - T \geq 0$. In order to see that $U \geq 0$, we need only note that for any finite set F ,

$$\begin{aligned} U \left(\mathbf{1} - \sum_{n \in F} \mathbf{e}_n \right) &= y - \sum_{n \in F} y_n \\ &= y - \bigvee_{n \in F} y_n \quad \text{as the } y_n \text{ are disjoint} \\ &\geq 0 \end{aligned}$$

then invoke Proposition 2.2. Thus $T \in L^r(\ell_0^\infty(I), F)$, which is assumed to be a Riesz space, so that T^+ exists. Again we see that $T^+ \mathbf{1} \geq T^+ \mathbf{e}_i \geq T \mathbf{e}_i = y_i$ for each $i \in I$, so that $T^+ \mathbf{1}$ is an upper bound for $\{y_i : i \in I\}$. On the other hand $T^+ \mathbf{1} \leq U \mathbf{1} = y$, where y was any upper bound for that set, so $T^+ \mathbf{1}$ must be its supremum. ■

What else is there to say about Dedekind completeness? If we are considering all Riesz spaces, not a lot. However if we were restricting our attention to, say Banach lattices F , then we would want to restrict E to being a Banach lattice. However our spaces $\ell_0^\infty(I)$ are clearly not Banach lattices. It is however possible to modify our proofs to work in that setting. In fact the proof is rather easier as Banach lattices are always uniformly complete and therefore we can use the criterion for Dedekind completeness given in Corollary 1.7 immediately. In order to state the best possible result here we will include an equivalence that we will not be in a position to prove for some time.

Theorem 2.5. *If F is a Banach lattice then the following are equivalent.*

- (i) *F is Dedekind complete.*
- (ii) *For all Banach lattices E , $L^r(E, F)$ is a Dedekind complete Riesz space.*
- (iii) *For all Banach lattices E , $L^r(E, F)$ is a Riesz space.*
- (iv) *For all Banach lattices E , $L^r(E, F)$ has the Riesz Separation Property.*

Proof. Again we refer the reader elsewhere, such as [7, Theorem 1.3.2], for a proof that (i) $\Rightarrow$ (ii), and note that (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) is obvious. The proof will not be completed until §4 when we prove that (iv) $\Rightarrow$ (i), but for now we show that our existing techniques allow us to prove that (iii) $\Rightarrow$ (i). By Corollary 1.7, in order to prove that F is Dedekind complete it suffices to show that any disjoint set $\{y_i : i \in I\}$ of positive elements in F which is bounded above by y must have a supremum.

Take for E the space of all continuous real valued functions on the one point compactification of I , considered with its discrete topology. We write ∞ for the extra point of that compactification. We define a linear operator $T : E \rightarrow F$ by

$$Tf = \sum_{i \in I} (f(i) - f(\infty))y_i.$$

This sum, which actually only has countably many non-zero terms, converges relatively uniformly with respect to y and hence converges in the norm of E . Let us also define $U : E \rightarrow F$ by $Uf = Tf + f(\infty)y$ which is again clearly linear. If $f \geq 0$ then

$$(U - T)f = f(\infty)y \geq 0$$

and for each $i \in I$ we have $f(\infty)(y - y_i) + f(i)y_i \geq 0$ so that $f(\infty)y \geq (f(\infty) - f(i))y_i$. As the y_i are disjoint, we also have $f(\infty)y \geq \sum_{i \in I} (f(\infty) - f(i))y_i = -Tf$ so that $Uf \geq 0$. Thus $U \geq T, 0$ and hence $T \in L^r(E, F)$.

Note that $U \mathbf{1} = y$. If we suppose that T^+ exists in $L^r(E, F)$ then

$$T^+ \mathbf{1} \geq T^+ \mathbf{e}_i \geq T \mathbf{e}_i = y_i$$

so that $T^+ \mathbf{1}$ is an upper bound for the set $\{y_i : i \in I\}$. On the other hand $T^+ \leq U$ so that $T^+ \mathbf{1} \leq U \mathbf{1} = y$. But y was any upper bound of the set $\{y_i : i \in I\}$, so that $T^+ \mathbf{1}$ is actually the supremum of the set $\{y_i : i \in I\}$, and hence F is indeed Dedekind complete. ■

If we specialise our spaces F even more then other characterisations become available to us. We can start including conditions on the norm of operators to weaken the order theoretic requirements on the range, provided we assume that the range space is of type $C(K)$.

Theorem 2.6. *If K is a compact Hausdorff space then the following are equivalent.*

- (i) $C(K)$ is Dedekind complete.
- (ii) For all Banach lattices E , $L^r(E, C(K))$ is a Dedekind complete Banach lattice under the operator norm.
- (iii) For all compact Hausdorff spaces X , $L^r(C(X), C(K))$ is a Dedekind complete Banach lattice under the operator norm.
- (iv) For all Banach lattices E and all bounded linear operators $S : E \rightarrow C(K)$ there is $T : E \rightarrow C(K)$ with $T \geq S, -S$ and $\|T\| < 2\|S\|$.
- (v) For all compact Hausdorff spaces X and all bounded linear operators $S : C(X) \rightarrow C(K)$ there is $T : C(X) \rightarrow C(K)$ with $T \geq S, -S$ and $\|T\| < 2\|S\|$.

Proof. Again the proof that (i) $\Rightarrow$ (ii) is classical, an exposition being contained in Theorem 1.5.11 of [7]. Specialising (ii) to the case that $E = C(X)$ gives us (iii). We see that (ii) $\Rightarrow$ (iv) and that (iii) $\Rightarrow$ (v) by taking $T = |S|$, and we also see that specialising (iv) to the case $E = C(X)$ gives us (v). It remains only to prove that (v) $\Rightarrow$ (i).

We must prove that K is Stonean. If not, let U_1, U_2 be disjoint open sets in K with $k \in \overline{U_1} \cap \overline{U_2}$. Let Σ_j denote the one-point compactification of U_j , with the extra point being denoted by σ_j . Define $S_j : C(\Sigma_j) \rightarrow C(K)$ by

$$S_j f = \begin{cases} f - f(\sigma_j) & \text{on } U_j \\ 0 & \text{on } K \setminus U_j \end{cases}$$

then S_j is linear with norm 2 (using supremum norms on all spaces involved). If we norm the product $C(\Sigma_1) \times C(\Sigma_2)$ by defining $\|(f_1, f_2)\| = \max\{\|f_1\|, \|f_2\|\}$ then the linear operator $S : C(\Sigma_1) \times C(\Sigma_2) \rightarrow C(K)$, defined by $S(f_1, f_2) = S_1 f_1 + S_2 f_2$, also has norm 2, since $S_1 f_1$ and $S_2 f_2$ are disjoint. Note that we may identify $C(\Sigma_1) \times C(\Sigma_2)$ with $C(X)$ where X is the compact Hausdorff space which is a disjoint union of Σ_1 and Σ_2 .

If $u \in U_j$ then we may find $f_u \in C(\Sigma_j)$, such that $-1 \leq f_u \leq 1$, $f_u(u) = 1$ and $f_u(\sigma_j) = -1$. Now, if $T_j : C(\Sigma_j) \rightarrow C(K)$ has $T_j \geq S_j, -S_j$ then we have

$$T_j \mathbf{1}_{\Sigma_j} - S_j f_u = (1/2) [(T_j + S_j)(\mathbf{1}_{\Sigma_j} - f_u) + (T_j - S_j)(\mathbf{1}_{\Sigma_j} + f_u)] \geq 0,$$

so that $T_j \mathbf{1}_{\Sigma_j}(u) \geq f_u(u) = 1$. Hence $T_j \mathbf{1}_{\Sigma_j} \geq 1$ on U_j and hence on $\overline{U_j}$. In particular $T_j \mathbf{1}_{\Sigma_j}(k) \geq 1$.

If $T : C(\Sigma_1) \times C(\Sigma_2) \rightarrow C(K)$ and $T \geq S, -S$ then defining $T_j = T|_{C(\Sigma_j)}$ defines an operator T_j that dominates S_j and $-S_j$. Thus

$$T(\mathbf{1}_{\Sigma_1}, \mathbf{1}_{\Sigma_2})(k) = T_1 \mathbf{1}_{\Sigma_1}(k) + T_2 \mathbf{1}_{\Sigma_2}(k) \geq 2,$$

whilst $\|(\mathbf{1}_{\Sigma_1}, \mathbf{1}_{\Sigma_2})\| = 1$, and we see that $\|T\| \geq 2$. Given that $\|S\| = 1$, this shows that (v) fails. ■

3. Characterisations of Dedekind σ -completeness.

In many senses, the results for Dedekind σ -completeness are simpler than those in the preceding section. The first result is somewhat typical.

Theorem 3.1. *If F is a Riesz space then the following are equivalent.*

- (i) F is Dedekind σ -complete.
- (ii) $L^r(\ell_0^\infty(\mathbb{N}), F)$ is a Dedekind σ -complete Riesz space.
- (iii) $L^r(\ell_0^\infty(\mathbb{N}), F)$ is a Riesz space.

Proof. That (ii) $\Rightarrow$ (iii) is clear, whilst the proof that (iii) $\Rightarrow$ (i) is the first part of the proof of Theorem 2.2. It only remains to prove that (i) $\Rightarrow$ (ii). We defer the proof of this until after we have proved the next result. ■

It is possible to prove a rather more useful result in the Banach lattice setting as then we have continuity to help us.

Theorem 3.2. *For a Banach lattice F the following are equivalent:*

- (i) *F is Dedekind σ -complete.*
- (ii) *For all separable Banach lattices E , $L^r(E, F)$ is a Dedekind σ -complete Riesz space.*
- (iii) *For all separable Banach lattices E , $L^r(E, F)$ is a Riesz space.*
- (iv) *$L^r(c, F)$ is a Riesz space.*

Proof. . If $x \in E_+$ then we may find a countable dense subset $A \subseteq [-x, x]$. If $T \in L^r(E, F)$ then T is certainly order bounded, so the set $T([-x, x])$ is bounded above in F . In particular, the countable set $T(A)$ is bounded above and has a supremum, Sx , if we assume that (i) holds. The density of A in $[-x, x]$ and the continuity of T means that Sx is also the supremum of the set $T([-x, x])$. Now the proof given in [7], Theorem 1.3.2, may be repeated (with obvious modifications) to complete the proof that (i) $\Rightarrow$ (ii). It is clear that (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv). In order to prove that (iv) $\Rightarrow$ (i), we repeat the argument used to prove (iii) $\Rightarrow$ (i) in Theorem 2.5, except that we only apply it to countable disjoint order bounded sets, when the space E is simply c . ■

Remainder of proof of Theorem 3.1. If $T \in L^r(\ell_0^\infty(\mathbb{N}), F)$ and F is Dedekind σ -complete, pick $y \in F$ such that $T[-1, 1] \subseteq [-y, y]$, so that $T\ell_0^\infty(\mathbb{N}) \subseteq F_y$, since $\mathbf{1}$ is an order unit for $\ell_0^\infty(\mathbb{N})$. Consider now T as an element of $L^r(\ell_0^\infty(\mathbb{N}), F_y)$, where we make F_y a Banach lattice by giving it the norm $\|\cdot\|_y$ induced by y . The proof of Theorem 3.2 remains valid even though $\ell_0^\infty(\mathbb{N})$ is not norm complete. Thus we may compute T^+ in $L^r(\ell_0^\infty(\mathbb{N}), F_y)$. Clearly, when regarded as an operator into F , T^+ is still an upper bound for T and 0 . On the other hand, if $S \in L^r(\ell_0^\infty(\mathbb{N}), F)$ and $S \geq T, 0$ then consider a dense countable subset, A , of $[0, x]$ for a given $0 \leq x \in \ell_0^\infty(\mathbb{N})$. We know that T^+ , as computed in $L^r(\ell_0^\infty(\mathbb{N}), F_y)$, maps x to $\bigvee T[0, x] = \bigvee T(A)$. But if $a \in [0, x]$ then $Sx \geq Sa \geq Ta$ so that certainly

$$Sx \geq \bigvee T(A) = \bigvee T[0, x] = T^+x.$$

I.e. $S \geq T^+$, so that T^+ is also the positive part of T in $L^r(\ell_0^\infty(\mathbb{N}), F)$, which is therefore a lattice. The proof of Dedekind σ -completeness proceeds as does the proof of Dedekind completeness in the proof of Theorem 1.3.2 of [7]. ■

After reading §2, a reader's expectation might well be that we were now going to give a characterisation of Dedekind σ -complete $C(K)$ spaces which involved spaces of regular operators with separable domain being Banach lattices plus a possible equivalence involving upper bounds for $S, -S$ which were not too big. We do indeed have a result that involves that last kind of criterion, but it turns out not to characterise Dedekind σ -completeness! We will meet that result in the next section. Of course if $L^r(E, C(K))$ is a Banach lattice for all separable Banach lattices E then, in particular, it must be a lattice, so that Theorem 3.2 tells us that $C(K)$ is Dedekind σ -complete. Thus there is actually nothing new to be said in this case.

4. Characterisations of the countable interpolation property.

Our first result in this setting will be for Banach lattices. It may be regarded as an analogue of Theorem 3.2. Before stating and proving that result, we need some technical results—first a result on $C(K)$ -spaces and then one on the existence of operators on spaces of the form $C(\gamma(I))$.

Proposition 4.1. *Let K be a compact Hausdorff space and A be an open F_σ -subset of K . There are two sequences in the space $C(K)$, of all continuous real-valued functions on K , (f_n) and (g_n) , with the following properties:*

- (i) $0 \leq f_n(k), g_n(k) \leq 1$ for all $k \in K$.
- (ii) If $m \neq n$ then $f_m \wedge f_n = g_m \wedge g_n = 0$.
- (iii) $f_n \wedge g_n = 0$ for all $n \in \mathbb{N}$.
- (iv) $\bigcup_{n=1}^\infty (f_n^{-1}(1) \cup g_n^{-1}(1)) = \bigcup_{n=1}^\infty \{k : f_n(k) > 0\} \cup \{k : g_n(k) > 0\} = A$.

Proof. Suppose that $A = \bigcup_{n=1}^\infty H_n$ where each H_n is closed. By the Tietze extension theorem, for each $n \in \mathbb{N}$ there is $p_n \in C(K)$ with

- (α) $0 \leq p_n(k) \leq 1$ for all $k \in K$,
- (β) $p_n(k) = 1$ for all $k \in H_n$, and

(γ) $p_n(k) = 0$ for all $k \notin A$.

If we define $p = \sum_{n=1}^{\infty} 2^{-n} p_n$ then this series is uniformly convergent so that $p \in C(K)$ and it is clear that $0 \leq p(k) \leq 1$ for all $k \in K$ and that $p(k) > 0$ if and only if $k \in A$. Let (α_n) be a strictly decreasing sequence of positive reals converging to zero, with $\alpha_1 = \sup p(K)$. Let $F_n = p^{-1}([\alpha_{2n}, \alpha_{2n-1}])$ for each $n \in \mathbb{N}$, so that each F_n is closed and $F_n \cap F_m = \emptyset$ if $m \neq n$. Let also $U_n = p^{-1}((\frac{\alpha_{2n} + \alpha_{2n+1}}{2}, \frac{\alpha_{2n-1} + \alpha_{2n-2}}{2}))$ (setting $\alpha_0 = \alpha_1 + 1$, for example), so that each U_n is open, $F_n \subseteq U_n$ and we still have $U_m \cap U_n = \emptyset$ if $m \neq n$. We may use Urysohn's lemma to produce $f_n \in C(K)$ with

- (a) $0 \leq f_n(k) \leq 1$ for all $k \in K$,
- (b) $f_n(k) = 1$ for all $k \in F_n$, and
- (c) $f_n(k) = 0$ for all $k \notin U_n$.

The sequence (f_n) certainly does all that is claimed in (i) and (ii) of the statement of the theorem. Similarly working with the closed sets $G_n = p^{-1}([\alpha_{2n+1}, \alpha_{2n}])$ and the open sets $V_n = p^{-1}((\frac{\alpha_{2n+1} + \alpha_{2n+2}}{2}, \frac{\alpha_{2n} + \alpha_{2n-1}}{2}))$ we may find $g_n \in C(K)$ with

- (a') $0 \leq g_n(k) \leq 1$ for all $k \in K$,
- (b') $g_n(k) = 1$ for all $k \in G_n$, and
- (c') $g_n(k) = 0$ for all $k \notin V_n$.

Again the sequence (g_n) does all that is claimed in (i) and (ii).

Claim (iv) in the theorem is because

$$\begin{aligned} A &\supseteq \bigcup_{n=1}^{\infty} \{k : f_n(k) > 0\} \cup \{k : g_n(k) > 0\} \\ &\supseteq \bigcup_{n=1}^{\infty} (f_n^{-1}(1) \cup g_n^{-1}(1)) \\ &\supseteq \bigcup_{n=1}^{\infty} (F_n \cup G_n) \\ &\supseteq p^{-1}((0, \infty)) \quad (\text{as } \alpha_n \downarrow 0) \\ &= A. \end{aligned}$$

This only leaves claim (iii) to be verified. Note that $V_n \cap U_m = \emptyset$, and hence $f_m \wedge g_n = 0$, unless $m = n$ or $m = n + 1$. Adding two zero functions at the start of the sequence (g_m) does not alter what we have already established and ensures that f_n will be disjoint from g_n . ■

Proposition 4.2. *Let E be a Banach lattice, let I be a non-empty set with the discrete topology and let $\gamma(I)$ denote its one-point compactification. Suppose also that S, T be continuous linear operators from $C(\gamma(I))$ into E . Let $(y_i)_{i \in I}$ be a family in E with $S\mathbf{e}_i \leq y_i \leq T\mathbf{e}_i$ for all $i \in I$ and let $y \in E$ be arbitrary. There is a continuous linear operator R from $C(\gamma(I))$ into E with $R\mathbf{e}_i = y_i$ for all $i \in I$ and $R\mathbf{1} = y$.*

Proof. It clearly suffices to consider only the case that $S = 0$. We only need to verify that if $\alpha_i \rightarrow 0$ (for some real family (α_i)) then $\sum \alpha_i y_i$ converges in E . It also clearly suffices to prove this in the case that $\alpha_i \geq 0$ for all $i \in I$. If $\epsilon > 0$ there is a finite set $F \subset I$ such that for any finite set $G \subset I$ with $F \cap G = \emptyset$ we have

$$\left\| \sum_{i \in G} \alpha_i \mathbf{e}_i \right\| < \epsilon$$

and hence

$$\left\| \sum_{i \in G} \alpha_i T\mathbf{e}_i \right\| < \|T\|\epsilon.$$

But

$$0 \leq \sum_{i \in G} \alpha_i y_i \leq \sum_{i \in G} \alpha_i T\mathbf{e}_i$$

so that

$$\left\| \sum_{i \in G} \alpha_i y_i \right\| \leq \left\| \sum_{i \in G} \alpha_i T \mathbf{e}_i \right\| < \|T\| \epsilon$$

and hence $\sum \alpha_i y_i$ is Cauchy and therefore convergent. ■

Theorem 4.3. *The following conditions on a Banach lattice F are equivalent:*

- (i) *F has the countable interpolation property.*
- (ii) *$L^r(c, F)$ has the strong countable interpolation property.*
- (iii) *$L^r(c, F)$ has the Riesz Separation Property.*

Proof. Our first step in the proof will be to show that if F has the countable interpolation property then $L^r(c, F)$ has the strong countable interpolation property. To this end, let $(S_p), (U_p)$ be two sequences in $L^r(c, F)$ with

$$S_p \leq U_q \quad \forall p, q \in \mathbb{N} \quad (a)$$

and recall, Proposition 1.10, that these operators are all norm bounded. We wish to find a bounded linear operator R from c into F with

$$S_p \leq R \leq U_p \quad \forall p \in \mathbb{N} \quad (b)$$

and it will certainly be the case then that R is regular. By Proposition 2.3, (a) is equivalent to

$$S_p \mathbf{e}_n \leq U_q \mathbf{e}_n \quad \forall n, p, q \in \mathbb{N} \quad (a_1)$$

and

$$S_p(\mathbf{1} - \sum_{k=1}^n \mathbf{e}_k) \leq U_q(\mathbf{1} - \sum_{k=1}^n \mathbf{e}_k) \quad \forall n, p, q \in \mathbb{N} \quad (a_2)$$

Similarly the operator R that we wish to construct must satisfy the following two conditions

$$S_p \mathbf{e}_n \leq R \mathbf{e}_n \leq U_p \mathbf{e}_n \quad \forall n, p \in \mathbb{N} \quad (b_1)$$

and

$$S_p(\mathbf{1} - \sum_{k=1}^n \mathbf{e}_k) \leq R(\mathbf{1} - \sum_{k=1}^n \mathbf{e}_k) \leq U_p(\mathbf{1} - \sum_{k=1}^n \mathbf{e}_k) \quad \forall n, p \in \mathbb{N} \quad (b_2)$$

The countable interpolation property for F certainly tells us that for each $n \in \mathbb{N}$ we can find elements of F lying above $S_p \mathbf{e}_n$ and below $U_p \mathbf{e}_n$ for each $p \in \mathbb{N}$. Pick one such element and call it $R \mathbf{e}_n$. The sequence $(R \mathbf{e}_n)$ will be norm bounded and for any choice of $y = R \mathbf{1} \in F$ there will be a (unique) continuous linear extension of R to the whole of c by Proposition 4.2. Thus we need only worry about choosing $R \mathbf{1}$ so that (b_2) holds. This is equivalent to asking that

$$S_p(\mathbf{1} - \sum_{k=1}^n \mathbf{e}_k) + \sum_{k=1}^n R \mathbf{e}_k \leq R \mathbf{1} \leq U_p(\mathbf{1} - \sum_{k=1}^n \mathbf{e}_k) + \sum_{k=1}^n R \mathbf{e}_k \quad \forall n, p \in \mathbb{N} \quad (b_3)$$

which we will be able to do, using the countable interpolation property, if we can show that

$$S_p(\mathbf{1} - \sum_{k=1}^n \mathbf{e}_k) + \sum_{k=1}^n R \mathbf{e}_k \leq U_q(\mathbf{1} - \sum_{k=1}^n \mathbf{e}_k) + \sum_{k=1}^n R \mathbf{e}_k \quad \forall n, p, q \in \mathbb{N}. \quad (b_4)$$

Let us now introduce a further sequence of elements V_m of $L^r(c, F)$ defined by requiring that

$$V_m(\mathbf{e}_n) = \begin{cases} S_1(\mathbf{e}_n) & \text{if } m \neq n \\ R \mathbf{e}_n & \text{if } m = n \end{cases}$$

$$V_m \mathbf{1} = S_1 \mathbf{1}.$$

By Proposition 4.2 bounded linear operators, V_m , with such properties do exist. Note that if $m \neq n$ then

$$V_m \mathbf{e}_n = S_1 \mathbf{e}_n \leq U_q \mathbf{e}_n$$

whilst if $m = n$ then

$$V_m \mathbf{e}_n = R \mathbf{e}_n \leq U_q \mathbf{e}_n$$

so that we always have $V_m \mathbf{e}_n \leq U_q \mathbf{e}_n$. Also we have, if $m \leq n$,

$$(U_q - V_m) \left(\mathbf{1} - \sum_{k=1}^n \mathbf{e}_k \right) = (U_q - S_1) \left(\mathbf{1} - \sum_{k=1}^n \mathbf{e}_k \right) + (R \mathbf{e}_m - S_1 \mathbf{e}_m) \geq 0$$

since $U_q \geq S_1$ and $R \mathbf{e}_m \geq S_1 \mathbf{e}_m$, whilst if $m > n$ then certainly

$$(U_q - V_m) \left(\mathbf{1} - \sum_{k=1}^n \mathbf{e}_k \right) = (U_q - S_1) \left(\mathbf{1} - \sum_{k=1}^n \mathbf{e}_k \right) \geq 0.$$

By Proposition 2.3 we thus have $U_q \geq V_m$ for all $m, q \in \mathbb{N}$.

Consider the set $\mathcal{F} = \{S_p : p \in \mathbb{N}\} \cup \{V_m : m \in \mathbb{N}\}$ and the set $\mathcal{G}$ of all finite suprema, *calculated in* $L^r(c, F'')$, from $\mathcal{F}$. If $T_1, T_2 \in \mathcal{F}$ then for each $n \in \mathbb{N}$ we have, using the Riesz-Kantorovich formula ([7], Corollary 1.3.4), which we may as F'' is Dedekind complete, and using the fact that each $\mathbf{e}_n$ is an atom (i.e. if $0 \leq x \leq \mathbf{e}_n$ then there is a real number α_n such that $x = \alpha_n \mathbf{e}_n$),

$$\begin{aligned} (T_1 \vee T_2)(\mathbf{e}_n) &= \sup\{\lambda T_1 \mathbf{e}_n + (1 - \lambda) T_2 \mathbf{e}_n : 0 \leq \lambda \leq 1\} \\ &= T_1 \mathbf{e}_n \vee T_2 \mathbf{e}_n. \end{aligned}$$

From the definition of, first, $R \mathbf{e}_n$ and then of V_m we certainly have $T \mathbf{e}_n \leq R \mathbf{e}_n$ for all $T \in \mathcal{F}$ and hence for all $T \in \mathcal{G}$. If Q denotes the supremum of the family $\mathcal{G}$ in $L^r(c, F'')$ then, as $\mathcal{G}$ is upward directed, for each $x \in c_+$ we have $Qx = \bigvee \{Tx : T \in \mathcal{G}\}$, so in particular we have

$$Q \mathbf{e}_n = \bigvee \{T \mathbf{e}_n : T \in \mathcal{G}\} \leq R \mathbf{e}_n.$$

But we also must have $Q \mathbf{e}_n \geq V_n \mathbf{e}_n = R \mathbf{e}_n$ so that we actually have

$$Q \mathbf{e}_n = R \mathbf{e}_n \quad \forall n \in \mathbb{N} \tag{c}$$

We thus have found $Q \in L^r(c, F'')$ with $S_p \leq Q \leq U_q$ for all $p, q \in \mathbb{N}$ and also with $Q \mathbf{e}_n = R \mathbf{e}_n \in F$ for all $n \in \mathbb{N}$. If it weren't for the fact that $Q \mathbf{1}$ need not lie in F then we would be finished.

The fact that $S_p \leq Q \leq U_q$ tells us that

$$S_p \left(\mathbf{1} - \sum_{k=1}^n \mathbf{e}_k \right) \leq Q \left(\mathbf{1} - \sum_{k=1}^n \mathbf{e}_k \right) \leq U_q \left(\mathbf{1} - \sum_{k=1}^n \mathbf{e}_k \right) \quad \forall n, p, q \in \mathbb{N} \tag{d_1}$$

holds and hence, noting (c), that

$$S_p \left(\mathbf{1} - \sum_{k=1}^n \mathbf{e}_k \right) + \sum_{k=1}^n R \mathbf{e}_k \leq Q \mathbf{1} \leq U_q \left(\mathbf{1} - \sum_{k=1}^n \mathbf{e}_k \right) + \sum_{k=1}^n R \mathbf{e}_k \quad \forall n, p, q \in \mathbb{N} \tag{d_2}$$

In particular this shows that we do have

$$S_p \left(\mathbf{1} - \sum_{k=1}^n \mathbf{e}_k \right) + \sum_{k=1}^n R \mathbf{e}_k \leq U_q \left(\mathbf{1} - \sum_{k=1}^n \mathbf{e}_k \right) + \sum_{k=1}^n R \mathbf{e}_k \quad \forall n, p, q \in \mathbb{N}. \tag{d_3}$$

As the order in F'' extends that in F , this is precisely (b_4) so the proof of the first implication is complete.

The strong countable interpolation property clearly implies the RSP, so let us now suppose that $L^r(c, F)$ has the RSP. Clearly, $L^r(c, I)$ has the RSP for any principal ideal I in F . In order to prove that F has the countable interpolation property, it suffices to prove that every principal ideal in F has the countable interpolation property. Thus, using Theorem 1.2, we may restrict our attention to the case that $F = C(K)$ for some compact Hausdorff space K . In view of Theorem 1.8, we must prove that K is an F -space.

Let A and B be disjoint open F_σ 's in K . We must prove that they have disjoint closures. By Proposition 4.1 we can find disjoint non-negative sequences $(s_n), (t_n), (u_n)$ and (v_n) in $C(K)$, lying under the constantly one function, with

$$\bigcup_{n=1}^{\infty} (s_n^{-1}(1) \cup t_n^{-1}(1)) = \bigcup_{n=1}^{\infty} \{k : s_n(k) > 0\} \cup \{k : t_n(k) > 0\} = A$$

and

$$\bigcup_{n=1}^{\infty} (u_n^{-1}(1) \cup v_n^{-1}(1)) = \bigcup_{n=1}^{\infty} \{k : u_n(k) > 0\} \cup \{k : v_n(k) > 0\} = B$$

and $s_n \wedge t_n = u_n \wedge v_n = 0$ for all $n \in \mathbb{N}$.

Define operators $S, T : c \rightarrow C(K)$ with $S\mathbf{e}_n = s_n, S\mathbf{1} = 0, T\mathbf{e}_n = t_n$ and $T\mathbf{1} = 0$. The disjointness of the sequences (s_n) and (t_n) guarantees the existence of such operators. For example if $\alpha_n \rightarrow 0$ and m, n are such that $\|\sum_{k=n}^m \alpha_k \mathbf{e}_k\| < \epsilon$ then each $|\alpha_k| < \epsilon$ and hence

$$\left\| \sum_{k=n}^m \alpha_k s_k \right\| = \left\| \bigvee_{k=n}^m \alpha_k s_k \right\| = \bigvee_{k=n}^m \|\alpha_k s_k\| = \bigvee_{k=n}^m |\alpha_k| \cdot \|s_k\| < \epsilon.$$

Define also $U, V : c \rightarrow C(K)$ with $U\mathbf{e}_n = s_n + t_n + u_n, V\mathbf{e}_n = s_n + t_n + v_n$, and $U\mathbf{1} = V\mathbf{1} = \mathbf{1}_K$, where $\mathbf{1}_K$ denotes the constantly one function in $C(K)$. This time the fact that the operators U and V extend to the whole of c is proved by writing them as $S + T$ plus another operator for which the image of the $(\mathbf{e}_n)$ is a disjoint sequence. Clearly $S\mathbf{e}_n, T\mathbf{e}_n \leq U\mathbf{e}_n, V\mathbf{e}_n$ for all $n \in \mathbb{N}$. We also have, for each $n \in \mathbb{N}$,

$$\begin{aligned} (U - S)(\mathbf{1} - \sum_{k=1}^n \mathbf{e}_k) &= U\mathbf{1} - S\mathbf{1} + \sum_{k=1}^n S\mathbf{e}_k - \sum_{k=1}^n U\mathbf{e}_k \\ &= \mathbf{1}_K + 0 + \sum_{k=1}^n s_k - \sum_{k=1}^n (s_k + t_k + v_k) \\ &= \mathbf{1}_K - \sum_{k=1}^n (t_k + v_k) \\ &\geq 0 \end{aligned}$$

as all the functions t_k and v_k for $1 \leq k \leq n$ are disjoint, non-negative and lie below $\mathbf{1}_K$. Thus $S \leq U$ and similarly we can show that $S, T \leq U, V$. By hypothesis there is a linear operator $R : c \rightarrow C(K)$ with $S, T \leq R \leq U, V$. Note first that

$$S\mathbf{e}_n, T\mathbf{e}_n \leq R\mathbf{e}_n \leq U\mathbf{e}_n, V\mathbf{e}_n \quad \forall n \in \mathbb{N}$$

so that

$$s_n, t_n \leq R\mathbf{e}_n \leq s_n + t_n + u_n, s_n + t_n + v_n$$

and hence

$$s_n \vee t_n = s_n + t_n \leq R\mathbf{e}_n \leq s_n + t_n + (u_n \wedge v_n) = s_n + t_n$$

showing that $R\mathbf{e}_n = s_n + t_n$. At this stage, let us ask what we know about $R\mathbf{1}$. For each $n \in \mathbb{N}$

$$\begin{aligned} S(\mathbf{1} - \sum_{k=1}^n \mathbf{e}_k), T(\mathbf{1} - \sum_{k=1}^n \mathbf{e}_k) &\leq R(\mathbf{1} - \sum_{k=1}^n \mathbf{e}_k) \\ &= R\mathbf{1} - \sum_{k=1}^n (s_k + t_k) \\ &\leq U(\mathbf{1} - \sum_{k=1}^n \mathbf{e}_k), V(\mathbf{1} - \sum_{k=1}^n \mathbf{e}_k). \end{aligned}$$

That is,

$$-\sum_{k=1}^n s_k, -\sum_{k=1}^n t_k \leq R\mathbf{1} - \sum_{k=1}^n (s_k + t_k) \leq \mathbf{1}_K - \sum_{k=1}^n (s_k + t_k + u_k), \mathbf{1}_K - \sum_{k=1}^n (s_k + t_k + v_k)$$

so that

$$\sum_{k=1}^n s_k, \sum_{k=1}^n t_k \leq R\mathbf{1} \leq \mathbf{1}_K - \sum_{k=1}^n u_k, \mathbf{1}_K - \sum_{k=1}^n v_k.$$

If, for some $n \in \mathbb{N}$ and some $k \in K$ we have $s_n(k) = 1$ or $t_n(k) = 1$ then this shows us that $R\mathbf{1}(k) = 1$ as if, for example, $s_n(k) = 1$ then

$$1 \leq \sum_{k=1}^n s_k(k) \leq R\mathbf{1}(k) \leq \mathbf{1}_K(k) = 1,$$

and similarly $R\mathbf{1}(k) = 0$ if $u_n(k) = 1$ or $v_n(k) = 1$. That is, the function $R\mathbf{1}$, which is an element of $C(K)$, takes the value 1 on A and 0 on B . This certainly suffices to prove that $\overline{A} \cap \overline{B} = \emptyset$ and hence that K is an F -space. ■

We are now in a position to sketch a proof that we omitted earlier.

Remainder of the proof of Theorem 2.5. It remains only to prove that (iv) $\Rightarrow$ (i). In order to show that, it suffices to prove that each principal ideal in E is Dedekind complete and hence, once again, we need only consider the case that $E = C(K)$ for some compact Hausdorff space K . In order to prove that $C(K)$ is Dedekind complete, we must prove that any two disjoint open subsets of K have disjoint closures, using Theorem 1.5. The proof is very similar to the last part of the proof of Theorem 4.3, so we will omit many of the details.

Let A and B be two such disjoint open subsets of K and let Γ be an index set sufficiently large that there are disjoint collections of open F_σ 's, $\{A_\gamma : \gamma \in \Gamma\}$ and $\{B_\gamma : \gamma \in \Gamma\}$ with $A_\gamma \subseteq A$ and $B_\gamma \subseteq B$ for each $\gamma \in \Gamma$ and $A \subseteq \overline{\bigcup_{\gamma \in \Gamma} A_\gamma}$ and $B \subseteq \overline{\bigcup_{\gamma \in \Gamma} B_\gamma}$. In order to accomplish this it may be necessary to take some of the A_γ and B_γ to be empty. In that case the corresponding sequences of functions that we will define will all be zero but that will not affect the proof at all. For each $\gamma \in \Gamma$, construct sequences $(s_{n\gamma}), (t_{n\gamma}), (u_{n\gamma})$ and $(v_{n\gamma})$ as in the proof of Theorem 4.1. Let E denote the Banach lattice of all real-valued functions on the discrete topological space $\mathbb{N} \times \Gamma$ which tend to a limit at infinity. If we now use $\mathbf{e}_{\gamma n}$ to denote the function on $\mathbb{N} \times \Gamma$ that is zero except at (n, γ) , where it takes the value 1, and $\mathbf{1}$ to denote the constantly one function on $\mathbb{N} \times \Gamma$ then we may again construct operators $S, T, U, V : E \rightarrow C(K)$ with

$$\begin{aligned} S\mathbf{e}_{\gamma n} &= s_{\gamma n} & S\mathbf{1} &= 0 \\ T\mathbf{e}_{\gamma n} &= t_{\gamma n} & T\mathbf{1} &= 0 \\ U\mathbf{e}_{\gamma n} &= s_{\gamma n} + t_{\gamma n} + u_{\gamma n} & U\mathbf{1} &= 0 \\ V\mathbf{e}_{\gamma n} &= s_{\gamma n} + t_{\gamma n} + v_{\gamma n} & V\mathbf{1} &= 0 \end{aligned}$$

and then verify that $S, T \leq U, V$. If there is $R : E \rightarrow C(K)$ with $S, T \leq R \leq U, V$ then we will again have $R\mathbf{e}_{\gamma n} = s_{\gamma n} + t_{\gamma n}$ and we will again be able to see that $R\mathbf{1}$ will be constantly 1 on each A_γ and 0 on each

B_γ . Continuity will then tell us that $R\mathbf{1}$ is constantly 1 on A and 0 on B , showing that A and B do indeed have disjoint closures. ■

When we specialise to F being a $C(K)$ -space, it turns out that we are only able to obtain completely equivalent statements by restricting the domain to separable spaces of the same form. As is well-known, this is equivalent to asking that K be metrizable. Part of our proof works in the Banach lattice context, so we separate it out.

Theorem 4.4. *Let S be any compact metric space and F be an arbitrary Banach lattice which has the countable interpolation property. Every order bounded linear operator $T : C(S) \rightarrow F$ is regular.*

Proof. Let us denote by y_0 a fixed positive element in F satisfying $|Tx| \leq y_0$ for each $x \in C(S)$ with $\|x\| \leq 1$. Since F^{**} is Dedekind complete there exists the modulus $|T| : C(S) \rightarrow F^{**}$, and by the familiar Riesz-Kantorovich formula $|T|x = \sup\{Tx' : |x'| \leq x\}$ for any $x \geq 0$. In particular, $|T|x \leq y_0$ provided $\|x\| \leq 1$.

We will prove the result by a Zorn's lemma argument on the set of all pairs (E, U) , where E is a subspace of $C(S)$ containing the constants, and $U : E \rightarrow F$ is a linear operator with $U \geq |T|_E$ and $U\mathbf{1} = y_0$ (this last condition certainly forces that $\|U\| = \|U\mathbf{1}\| \leq \|y_0\|$). That this is actually a set (rather than a proper class) is routine to verify by cardinality arguments, and that it is non-empty follows by taking one-dimensional $E = \mathbb{R}\mathbf{1} = \{\alpha\mathbf{1} : \alpha \in \mathbb{R}\}$ and U to be the unique linear operator on E with $U\mathbf{1} = y_0$. As it may be expected we partially order this set by defining $(E_1, U_1) \succ (E_2, U_2)$ to mean that $E_1 \supseteq E_2$ and $U_1|_{E_2} = U_2$. It is routine to verify the conditions for using Zorn's lemma so that we have a maximal element (E, U) of this set. We claim that $E = C(S)$, so that U actually satisfies the requirements of the theorem.

If $E \neq C(S)$, let $g \in C(S) \setminus E$ and let $G = E \oplus \mathbb{R}g$ be the linear span of E and g . We show how to extend U to V on the whole of G with $V \geq |T|_G$ and $\|V\| \leq \|y_0\|$, contradicting the alleged maximality of (E, U) . Consider the sets $A = \{a \in E : a \geq g\}$ and $B = \{b \in E : b \geq -g\}$. As $C(S)$ is separable, both A and B are separable. Let A' and B' be countable dense subsets of A and B respectively. For $a \in A'$, let C_a be a countable dense subset of the set $\{u \in C(S) : |u| \leq a - g\}$. Similarly, for $b \in B'$, let D_b be a countable dense subset of the order interval $\{v \in C(S) : |v| \leq b + g\}$.

If $a \in A'$, $b \in B'$, $u \in C_a$ and $v \in D_b$, then $a - g \geq 0$ and $b + g \geq 0$ so that certainly $a + b \geq 0$. Hence

$$\begin{aligned} Ua + Ub &= U(a + b) \\ &\geq |T|(a + b) \\ &= |T|(a - g + b + g) \\ &= |T|(a - g) + |T|(b + g) \\ &\geq |T||u| + |T||v| \\ &\geq Tu + Tv \end{aligned}$$

so that

$$Ua - Tu \geq Tv - Ub.$$

This holds simultaneously for any choice of $a \in A'$, $u \in C_a$, $b \in B'$ and $v \in D_b$. Since there are only countably many such choices, we may use the countable interpolation property in F to find $h \in F$ with

$$Ua - Tu \geq h \geq Tv - Ub$$

for all $a \in A'$, $u \in C_a$, $b \in B'$ and $v \in D_b$.

Now let us define V on G by $V(f + \alpha g) = Uf + \alpha h$ for all $f \in E$ and $\alpha \in \mathbb{R}$. If $f \in B'$ then, for each $v \in D_f$, we have

$$V(f + g) = Uf + h \geq Tv$$

and by continuity this persists for all v with $|v| \leq f + g$. Hence, regarded as operators into F^{**} , we have

$$V(f + g) \geq |T|(f + g)$$

for all $f \in B'$. Again use the density of B' in B , together with the continuity of U , to see that we actually have

$$V(f + g) \geq |T|(f + g)$$

whenever $f + g \geq 0$. Similarly, if $f - g \geq 0$, then we also have

$$V(f - g) \geq |T|(f - g).$$

The linearity of V and of $|T|$ now make it clear that if $f + \alpha g \geq 0$, then $V(f + \alpha g) \geq |T|(f + \alpha g)$ whether $\alpha > 0$, $\alpha < 0$ or $\alpha = 0$ (in which case it is clear). It follows that $V \geq |T|_{|G}$, in particular V is positive. To finish the proof it remains to verify that $\|V\| \leq \|y_0\|$. Take any $z \in G$ with $\|z\| \leq 1$. Therefore $-\mathbf{1} \leq z \leq \mathbf{1}$ and, since V is positive, we have $\pm Vz \leq V\mathbf{1} = U\mathbf{1} = y_0$. This completes the proof. ■

Putting the last two results together we see that $L(c, F)$ will have the Riesz separation property when F is a Banach lattice with the countable interpolation property. What is missing from the preceding result is any indication of the norm of the operator U . We can provide this estimate only at the expense of restricting F even further.

Theorem 4.5. *If K is a compact Hausdorff space then the following are equivalent.*

- (i) $C(K)$ has the countable interpolation property.
- (ii) For all compact metric spaces S and $T \in L(C(S), C(K))$ there is $U \in L(C(S), C(K))$ with $U \geq T, -T$ and $\|U\| = \|T\|$.
- (iii) For all compact metric spaces S and $T \in L(C(S), C(K))$ there is $U \in L(C(S), C(K))$ with $U \geq T, -T$ and $\|U\| < 2\|T\|$.

Proof. In order to show that (i) $\Rightarrow$ (ii) we need to provide an estimate of the norm of U in the proof of Theorem 4.4. From the proof, the norm of U satisfies the estimate $\|U\| \leq \|y_0\|$, where y_0 is an arbitrary element in $C(K)_+$ majorizing the T -image of the unit ball in $C(S)$. But one such element is $y_0 = \|T\|\mathbf{1}_K$, with norm $\|T\|$, so we can ensure that $\|U\| \leq \|T\|$ under these circumstances.

Now we must prove that (iii) $\Rightarrow$ (i). If K is not an F -space, let G_1 and G_2 be disjoint open F_σ -sets in K with $k \in \overline{G_1} \cap \overline{G_2}$. We may write each G_j as a union $\bigcup_{m=1}^\infty F_j^m$, where each F_j^m is a compact set with $F_j^m \subseteq F_j^{(m+1)\circ}$. Let f_j^m be a continuous function on K such that $0 \leq f_j^m \leq \mathbf{1}$, $f_j^m|_{F_j^m} \equiv 1$ and $f_j^m|_{(F_{j+1}^m)^c} \equiv 0$. If Σ_j denotes the one point compactification of G_j , with σ_j denoting the point at infinity, then let E_j be the closed sublattice of $C(\Sigma_j)$ generated by the constants and the functions f_j^m . It is standard that E_j is a separable Banach lattice.

Let $E = E_1 \times E_2$ with the product norm, $\|(g_1, g_2)\| = \max\{\|g_1\|, \|g_2\|\}$. Define $T_j : E_j \rightarrow C(K)$ by

$$T_j g = \begin{cases} g - g(\sigma_j) & \text{on } G_j \\ 0 & \text{on } K \setminus G_j \end{cases}$$

and then define $T : E \rightarrow C(K)$ by $T(g_1, g_2) = T_1 g_1 + T_2 g_2$. As in the proof of Theorem 2.6 each T_j and T have norm 2.

If $U_j : E_j \rightarrow C(K)$ with $U_j \geq T_j, -T_j$ then, for each m , we have

$$U_j \mathbf{1}_{\Sigma_j} - T_j(2f_j^m - \mathbf{1}_{\Sigma_j}) = \frac{1}{2}[(U_j + T_j)(2 \cdot \mathbf{1}_{\Sigma_j} - 2f_j^m) + (U_j - T_j)(2f_j^m)] \geq 0.$$

It follows that $(U_j \mathbf{1}_{\Sigma_j})|_{F_j^m} \geq T_j(2f_j^m - \mathbf{1}_{\Sigma_j})|_{F_j^m} = 2$. By continuity, we must have $(U_j \mathbf{1}_{\Sigma_j})(k) \geq 2$.

If now we have $U : E \rightarrow C(K)$ with $U \geq T, -T$ then letting U_j denote the restriction of U to E_j we have $U_j \geq T_j, -T_j$. Hence $U((\mathbf{1}_{\Sigma_1}, \mathbf{1}_{\Sigma_2}))(k) = U_1 \mathbf{1}_{\Sigma_1}(k) + U_2 \mathbf{1}_{\Sigma_2}(k) \geq 4$, whilst $\|(\mathbf{1}_{\Sigma_1}, \mathbf{1}_{\Sigma_2})\| = 1$, so that $\|U\| \geq 4$. Given that $\|T\| = 2$, this establishes the result. ■

5. Characterisations of Dedekind α -completeness.

As well as Dedekind completeness, we have met the restricted notion of Dedekind σ -completeness which only demands that countable sets which are bounded above must have a supremum. If α is an infinite cardinal then a Riesz space is termed α -complete if every subset of cardinality at most α , which is bounded above, must have a least upper bound. One would expect to be able to prove an analogue of Theorem 3.1 in order to characterise Dedekind α -completeness but a cursory inspection of that proof shows that it depends on the ordering of the natural numbers in an essential manner. It is possible to generalise the argument to other cardinals, but more care is needed.

Theorem 5.1. *For a fixed cardinal number α the following conditions on a Riesz space F are equivalent:*

- (i) *Any subset of F of cardinality at most α , which has an upper bound, has a supremum.*
- (ii) *If η is an ordinal of cardinality at most α , $f : \eta \rightarrow F$ is an increasing function and $f(\eta)$ has an upper bound, then $f(\eta)$ has a supremum.*
- (iii) *If η is an initial ordinal of cardinality at most α , $f : \eta \rightarrow F$ is an increasing function and $f(\eta)$ has an upper bound, then $f(\eta)$ has a supremum.*
- (iv) *$L^b(\ell_0^\infty(\alpha), F)$ is a Riesz space.*
- (v) *$L^r(\ell_0^\infty(\alpha), F)$ is a Riesz space.*

Proof. . It is clear that (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii). In order to establish that (iii) $\Rightarrow$ (i), we may suppose that there is some subset of F with an upper bound but no supremum (else (i) is certainly true). Let β be the smallest cardinal of a set $B \subset F$ which is bounded above but has no supremum. Let η be the first ordinal of cardinality equal to β . Index B by the elements of η , so that $B = \{b_i : i < \eta\}$. For each $i < \eta$, $\text{card}(i) < \beta$, so that $f(i) = \sup\{b_j : j < i\}$ exists in F , by the definition of β . Now $f : \eta \rightarrow F$ is an increasing function which is bounded above. To establish (i) we need to prove that $\beta > \alpha$. If not, then $\beta \leq \alpha$ so that (iii) guarantees that $\sup f(\eta)$ exists. Clearly $\sup f(\eta) \geq f(i+1) \geq b_i$ for each $i \in \eta$, so that $\sup f(\eta)$ is an upper bound for B . On the other hand any upper bound c for B will also be an upper bound for each set $\{b_j : j < i\}$, so $c \geq f(i)$ and hence $c \geq \sup f(\eta)$. Thus $\sup f(\eta)$ is the supremum of B , contrary to hypothesis. Thus $\beta > \alpha$ and hence (i) holds.

In order to prove that (i) $\Rightarrow$ (iv), notice first that for every $x \in \ell_0^\infty(\alpha)_+$ we may find a subset A of the order interval $[0, x]$ of cardinality at most α , which is dense for the supremum norm. If $T : \ell_0^\infty(\alpha) \rightarrow F$ is order bounded then there will be $y \in F$ such that $T([-1, 1]) \subseteq [-y, y]$ so that $T(A)$ is relatively uniformly dense in $T([0, x])$ with respect to y . By (i) $T(A)$ has a supremum in F which must also be the supremum of $T([0, x])$. It is now routine to define T^+ on $\ell_0^\infty(\alpha)_+$ by $T^+x = \sup T([0, x])$ and extend it to a linear operator on the whole of $\ell_0^\infty(\alpha)$. The operator T^+ is the supremum of T and the zero operator showing that $L^b(\ell_0^\infty(\alpha), F)$ is a Riesz space.

Finally we will prove that (v) $\Rightarrow$ (iii). Before doing this we show that if (v) holds for α then it also holds for any infinite cardinal $\beta < \alpha$. We may suppose that $\beta < \alpha$ and define $J : \ell_0^\infty(\beta) \rightarrow \ell_0^\infty(\alpha)$ by extending elements of $\ell_0^\infty(\beta)$ to have a constant value on $\alpha \setminus \beta$ (the same value that they take on all but a finite number of points of β). We also have the restriction map $R : \ell_0^\infty(\alpha) \rightarrow \ell_0^\infty(\beta)$ and clearly $R \circ J$ is the identity on $\ell_0^\infty(\beta)$. Note that both R and J are positive. If $T \in L^r(\ell_0^\infty(\beta), F)$, then $T \circ R \in L^r(\ell_0^\infty(\alpha), F)$, so has a positive part $(T \circ R)^+$. Consider $(T \circ R)^+ \circ J \in L^r(\ell_0^\infty(\beta), F)$. Obviously this operator is positive. If $x \in \ell_0^\infty(\beta)_+$, then $Jx \geq 0$ so that $(T \circ R)^+(Jx) \geq (T \circ R)(Jx) = Tx$. Thus $(T \circ R)^+ \circ J$ is a positive majorant for T . If S is any other positive majorant for T , then $S \circ R \geq T \circ R, 0$ so that $S \circ R \geq (T \circ R)^+$, and hence $S = S \circ R \circ J \geq (T \circ R)^+ \circ J$, showing that $(T \circ R)^+ \circ J$ is actually the positive part of T and $L^r(\ell_0^\infty(\beta), F)$ is indeed a Riesz space.

Now suppose that (iii) fails. Let η be the initial ordinal of lowest cardinality for which it fails. Then $\beta = \text{card}(\eta) \leq \alpha$. In view of the preceding paragraph we know that $L^r(\ell_0^\infty(\eta), F)$ is a Riesz space. Let $f : \eta \rightarrow F$ be any increasing function for which $f(\eta)$ has an upper bound but no supremum. Without loss of generality we may suppose that $f(0) = 0$, the zero element in F . Define $T : \ell_0^\infty(\eta) \rightarrow F$ as follows

$$\begin{aligned} T\mathbf{1} &= 0 \\ T(\mathbf{e}_0) &= f(0) \\ T(\mathbf{e}_i) &= f(i) - \bigvee_{j < i} f(j), \quad i \in \eta \end{aligned}$$

This supremum exists as $\text{card}(i) < \beta$ and because if (3) holds with α replaced by $\text{card}(i)$ then so does (2). If c is any upper bound for $f(\eta)$ in F , then we may define $T_c : \ell_0^\infty(\eta) \rightarrow F$ by

$$\begin{aligned} T_c \mathbf{1} &= c \\ T_c(\mathbf{e}_i) &= T(\mathbf{e}_i). \end{aligned}$$

We claim that T_c is a positive majorant for T , thus showing that T is regular. If $x = \mathbf{1} + \sum_{k \in K} x_k \mathbf{e}_k \in \ell_0^\infty(\eta)_+$, where $K = \{k_1, k_2, \dots, k_n\}$ is a finite subset of η with $k_1 > k_2 > \dots > k_n$, then (noting that each $x_k \geq -1$ and that $T(\mathbf{e}_k) \geq 0$) we have

$$\begin{aligned} T_c(x) &= T_c \mathbf{1} + \sum_{k \in K} x_k T(\mathbf{e}_k) \\ &\geq T_c \mathbf{1} - \sum_{k \in K} T(\mathbf{e}_k) \\ &= c - [f(k_1) - \bigvee_{j < k_1} f(j) + f(k_2) - \bigvee_{j < k_2} f(j) + \dots + f(k_n) - \bigvee_{j < k_n} f(j)] \\ &\geq c - f(k_1) + \bigvee_{j < k_n} f(j) \\ &\geq c - f(k_1) \geq 0. \end{aligned}$$

Also, for the same x as above, we have

$$\begin{aligned} (T_c - T)(x) &= (T_c(\mathbf{1}) - T(\mathbf{1})) + \sum_{k \in K} (T_c(\mathbf{e}_k) - T(\mathbf{e}_k)) \\ &= c \geq f(0) = 0. \end{aligned}$$

Since any positive element of $\ell_0^\infty(\eta)$ is a positive multiple of such an x , it follows that $T_c \geq T, 0$ as claimed.

By hypothesis, T^+ exists in $L^r(\ell_0^\infty(\eta), F)$. If K is any finite subset of η then we have

$$\mathbf{1} \geq \sum_{k \in K} \mathbf{e}_k \geq 0$$

so that

$$T^+(\mathbf{1}) \geq T^+(\sum_{k \in K} \mathbf{e}_k) \geq \sum_{k \in K} T(\mathbf{e}_k).$$

We claim that if $i < \eta$, then any upper bound for all the sums $\sum_{k \in K} T(\mathbf{e}_k)$, where K is a finite set for which each $k \in K$ is at most i , must be at least $f(i)$. If not, let i_0 be the first ordinal for which this fails. Then $i_0 \neq 0$ as

$$\bigvee_{k \leq 0} T(\mathbf{e}_k) = T(\mathbf{e}_0) = f(0)$$

by definition. Otherwise, if u is an upper bound for all such sums $\sum_{k \in K} T(\mathbf{e}_k)$, with each $k \leq i_0$, then $u - T(\mathbf{e}_{i_0})$ will be an upper bound for all sums $\sum_{k \in K} T(\mathbf{e}_k)$, where each $k < i_0$. In particular, for each $j < i_0$, it will be an upper bound for the sums $\sum_{k \in K} T(\mathbf{e}_k)$ where each $k \leq j$. By definition of i_0 , any such upper bound is at least $f(j)$. Thus $u - T(\mathbf{e}_{i_0}) \geq f(j)$ for all $j < i_0$. That is, we must have

$$u \geq T(\mathbf{e}_{i_0}) + \bigvee_{j < i_0} f(j) = f(i_0).$$

In other words we have proved that $T^+(\mathbf{1}) \geq f(i)$ for all $i \in \eta$. For any upper bound c of $f(\eta)$, T_c is a positive majorant of T so that $T_c \geq T^+$. Thus $c = T_c(\mathbf{1}) \geq T^+(\mathbf{1})$ and $T^+(\mathbf{1})$ is the supremum of $f(\eta)$. This contradicts the choice of η and f , so that we indeed have (iii) holding. ■

6. Some gaps in our results.

There are, in the later parts of these results, some rather obvious omissions. Some of these are genuine open problems. Others are results that I am fairly sure are true, but where there is insufficient interest in the result to spend the time writing out a proof.

Problem 6.1. The first major gap is at the start of §4, where one would expect a characterisation of those Riesz spaces with the countable interpolation property—presumably in terms of the space of regular operators from $\ell_0^\infty(\mathbb{N})$ into it having the Riesz separation property. The proof of (iii) $\Rightarrow$ (i) in Theorem 4.3 will certainly not work in such a setting (although it would for uniformly complete Riesz spaces). Zafer Erçan has pointed out that (i) $\Rightarrow$ (iii) can be proved by localizing the problem to a principal ideal and that a version of (ii) involving the separation of countable order bounded sequences can be proved similarly.

Problem 6.2. An alternative method of proving (i) $\Rightarrow$ (ii) in Theorem 4.3 would be to use the kind of Zorn's lemma argument used in the proof of Theorem 4.4. This should allow one to prove that $L(C(S), F)$ has the Riesz separation property when S is a compact metric space and F is a Banach lattice with the countable interpolation property. The proof is likely to be quite lengthy, but *should* be a fairly routine modification of the proof of Theorem 4.4, incorporating some of ideas of our proof of Theorem 4.3. It is unlikely, however, that such a result would be publishable in a major journal.

Problem 6.3. Presumably it is possible to prove a variant of Theorem 5.1 in a Banach lattice setting. This should be fairly routine. It might also be worth looking at the case when $F = C(K)$.

Bibliography

This list contains three types of item. There are several books which contain a lot more information about Banach lattices and positive operators in general. There is also one paper that we have cited for a proof. Finally there are several papers by the author and/or his collaborator, Y.A. Abramovich, which contain more details on the results presented here and related material.

1. Y.A. ABRAMOVICH and V.A. GEJLER, On a question of Fremlin concerning order bounded and regular operators, *Coll. Math.* 46 (1982) 15–17.
2. Y.A. ABRAMOVICH and A.W. WICKSTEAD, Regular operators from and into a small Riesz space, *Indag. Math. N.S.* 2 (1991) 257–274.
3. Y.A. ABRAMOVICH and A.W. WICKSTEAD, The regularity of order bounded operators into $C(K)$ II, *Quart. J. Math. Oxford (2)* 44 (1993) 257–270.
4. Y.A. ABRAMOVICH and A.W. WICKSTEAD, Some characterisations of Dedekind α -completeness of a Riesz space, *Canad. Math. Bull.* 36 (1993) 3–7.
5. C.D. ALIPRANTIS and O. BURKINSHAW, *Positive Operators*, Academic Press, Orlando-New York-San Diego-London, 1985.
6. W.A.J. LUXEMBURG and A.C. ZAAANEN, *Riesz Spaces I*, North-Holland, Amsterdam-London, 1971.
7. P. MEYER-NIEBERG, *Banach Lattices*, Springer-Verlag, Berlin-Heidelberg-New York, 1991.
8. H.H. SCHAEFER, *Banach Lattices and Positive Operators*, Springer-Verlag, Berlin-Heidelberg-New York, 1974.
9. G.L. SEEVER, Measures on F-spaces, *Trans. Amer. Math. Soc.* 133 (1968) 267–280.
10. A.C. ZAAANEN, *Riesz Spaces II*, North-Holland, Amsterdam-London, 1983.
11. A.W. WICKSTEAD, The regularity of order bounded operators into $C(K)$, *Quart. J. Math. Oxford (2)* 41 (1990) 359–368.
12. A.W. WICKSTEAD, Spaces of operators with the Riesz Separation Property, *Indag. Math.* (to appear).