

OPERATORS ON BANACH LATTICES

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INTRODUCTION

If X and Y are Banach lattices then there are several spaces of linear operators between them that may be studied. $\mathcal{L}(X, Y)$ is the space of all *norm bounded* operators from X into Y . There is no reason to expect there to be any connection between the order structure of X and Y and that of $\mathcal{L}(X, Y)$. $\mathcal{L}^r(X, Y)$ is the space of *regular* operators, i.e. the linear span of the positive operators. This at least has the merit that when it is ordered by the cone of positive operators then that cone is *generating*. $\mathcal{L}^b(X, Y)$ is the space of *order bounded* operators, which are those that map order bounded sets in X to order bounded sets in Y . We always have $\mathcal{L}^r(X, Y) \subseteq \mathcal{L}^b(X, Y) \subseteq \mathcal{L}(X, Y)$ and both inclusions may be proper.

I will look at the following problems, which are a small selection of those that we might have considered, and attempt to give a snapshot survey of our state of knowledge of these problems at the time of writing.

- (1) When are all bounded operators regular?
- (2) What can we say about the order structure of $\mathcal{L}^r(X, Y)$?
- (3) When are all order bounded operators regular?
- (4) What kind of duality theory is there for regular operators?
- (5) What happens for the classical Banach lattices?

I am not going to attempt to give complete proofs of these results, simply giving some of the easier proofs that give a flavour of the subject. More detailed references and some of the proofs may be found in [18].

1. WHEN ARE ALL BOUNDED OPERATORS REGULAR?

The reason this question is phrased this way is because every regular operator must be bounded:

Theorem 1.1. *If X is a Banach lattice and Y any normed lattice then $\mathcal{L}^r(X, Y) \subseteq \mathcal{L}(X, Y)$.*

Proof. It suffices to prove that every positive operator $T : X \rightarrow Y$ is norm bounded. If this is not the case then for all $n \in \mathbb{N}$ there is $x_n \in X$

with $\|x_n\| = 1$ and $\|Tx_n\| \geq n^3$. As X is a Banach lattice, $\| |x_n| \| = \|x_n\| = 1$. The positivity of T and the fact that $|x_n| \geq x_n \geq -|x_n|$ forces $T|x_n| \geq Tx_n \geq -T|x_n|$, so that $T|x_n| \geq |Tx_n|$ which in turn ensures that $\|T|x_n|\| \geq \|Tx_n\| \geq n^3$. Now consider the element $y = \sum_{n=1}^{\infty} |x_n|/n^2 \in X$ which exists as the series is absolutely convergent and X is norm complete. For each $m \in \mathbb{N}$ we have $y \geq |x_m|/m^2 \geq 0$ so that $Ty \geq T|x_m|/m^2 \geq 0$ and hence $\|Ty\| \geq \|T|x_m|/m^2\| \geq m$. This cannot hold for all $m \in \mathbb{N}$, so we have a contradiction and must admit that T is norm bounded. $\square$

And, of course, the converse is false. Recall that the Rademacher functions are defined on $[0, 1]$ by the formula $r_n(x) = \text{sign}(\sin(2^n \pi x))$, so that $|r_n| = r_0 \equiv 1$ and

$$\int_0^1 r_m(x)r_n(x) dx = \begin{cases} 1 & m \neq n \\ 0 & m = n. \end{cases}$$

Example 1.2. *There is a bounded linear operator $T : L_1[0, 1] \rightarrow c_0$ which is not regular.*

Proof. Let (r_n) denote the sequence of Rademacher functions on $[0, 1]$. As these lie in $L_\infty[0, 1]$ they may be regarded as elements of $L_1[0, 1]^*$ and there we have $r_n \rightarrow 0$ weak*. Define an operator $T : L_1[0, 1] \rightarrow c_0$ by $Tx = \sum_{n=1}^{\infty} r_n(x)e_n$ where e_n denotes the n 'th standard basis vector in c_0 . T is bounded as $\|Tx\|_\infty = \sup_{n=1}^{\infty} |r_n(x)| \leq \sup_{n=1}^{\infty} \|r_n\|_\infty \|x\|_1 = \|x\|_1$. Note that $T(r_n) = e_n$ for all $n \in \mathbb{N}$ and that $T(r_0) = 0$ so that $T(r_0 + r_n) = e_n$. As r_0 is constantly one, we see that $r_0 + r_n$ is positive. It follows that if we had $U \geq T, 0$ then

$$\begin{aligned} U(2r_0) &\geq U(r_0 + r_n) \\ &\geq T(r_0 + r_n) \\ &= e_n \end{aligned}$$

so that $U(2r_0) \geq e_n$ for all $n \in \mathbb{N}$, which is inconsistent with $U(2r_0)$ lying in c_0 . It follows that T is not regular after all. $\square$

It is thus a non-trivial and interesting question to ask when all the bounded operators from X to Y are regular. With most of the problems we shall consider concerning pairs of Banach lattices we would like to characterize the pairs (X, Y) for which a certain property holds. That is often not possible. What often *is* possible is to characterize those range spaces Y such that the desired property holds for all pairs (X, Y) and those domains X such that the property holds for all pairs (X, Y) . We start by considering such properties in this setting. What about the

nicest possible range spaces first of all? Before stating the result, let us define a class of ordered vector spaces that is slightly more general than vector lattices.

Definition 1.3. An ordered vector space X has the *Riesz Separation Property* (RSP) if whenever $x_1, x_2, z_1, z_2 \in X$ with $x_1, x_2 \leq z_1, z_2$ there is $y \in X$ with $x_1, x_2 \leq y \leq z_1, z_2$. It is equivalent that X has the *Riesz Decomposition Property* (RDP) which states that if $0 \leq y \leq x_1 + x_2$ and $x_1, x_2 \geq 0$ then there are $y_1, y_2 \geq 0$ such that $y = y_1 + y_2$ and $0 \leq y_k \leq x_k$ ($k = 1, 2$). This is equivalent to the statement that for all $x_1, x_2 \in X_+$ then $[0, x_1] + [0, x_2] = [0, x_1 + x_2]$.

Vector lattices necessarily have the Riesz Separation Property. The space $\{(x_n) : 2x_n \rightarrow x_1 + x_2\}$ has the Riesz Separation Property but is not a lattice.

Theorem 1.4. *If Y is a Dedekind complete Banach lattice then the following are equivalent:*

- (1) *Y has a strong order unit.*
- (2) *For all Banach lattices X and Y , $\mathcal{L}(X, Y) = \mathcal{L}^r(X, Y)$.*

Proof. The fact that (1) $\Rightarrow$ (2) dates back to Kantorovich's work in [8]. The proof shows that if $T : X \rightarrow Y$ is norm bounded then T^+ exists in $\mathcal{L}(X, Y)$. As T is norm bounded, if $x \in X_+$ then the set $T([0, x])$ is norm bounded (order intervals in Banach lattices are norm bounded) so has a supremum in Y which we denote by Sx . The map $x \mapsto Sx$ is additive on X_+ , because the RDP and linearity of T tells us that $T([0, x_1]) + T([0, x_2]) = T([0, x_1 + x_2])$, and positively homogeneous so that it extends to a linear operator on X which is easily seen to be the modulus of T .

The proof that (2) $\Rightarrow$ (1) is rather indirect and may be found in [3]. □

In this case we have $T^+(x) = \sup T([0, x])$ if $x \in X_+$. Formulae like this are referred to as *Riesz-Kantorovich* formulae.

Without the hypothesis of Dedekind completeness the implication (1) $\Rightarrow$ (2) fails. For example, Lemma 2.10 of [19] shows that if K is a compact Hausdorff space in which there is a sequence with no convergent subsequence then $\mathcal{L}(C(K), c) \neq \mathcal{L}^r(C(K), c)$. Similarly, condition (2) alone is not enough to force Dedekind completeness of Y , as is shown by an example in [2]. However we do have:

Theorem 1.5. *If Y is any Banach lattice then the following are equivalent:*

- (1) *Y is Dedekind complete with a strong order unit.*

- (2) *For all Banach lattices X , $\mathcal{L}(X, Y)$ is a vector lattice.*

The dual result is rather simpler to establish.

Theorem 1.6. *If X is a Banach lattice then the following are equivalent:*

- (1) *X is isomorphic to an atomic AL-space.*
- (2) *For all Banach lattices Y , $\mathcal{L}(X, Y) = \mathcal{L}^r(X, Y)$.*
- (3) *For all Banach lattices Y , $\mathcal{L}(X, Y)$ is a vector lattice.*

In many cases that are known when $\mathcal{L}(X, Y) = \mathcal{L}^r(X, Y)$ then either X is an AL-space or Y is an AM-space, at least up to isomorphism. That this is not always the case was established in an example due to Abramovich in [1], where he produced Banach lattices X and Y such that X is not isomorphic to an AL-space and Y is not isomorphic to an AM-space yet $\mathcal{L}(X, Y) = \mathcal{L}^r(X, Y)$.

Given that there is no complete characterization of exactly when all norm bounded operators are regular, and as the “universal domains” and “universal range spaces” are so specialized, it is worth seeking extra cases where we have equality. We will look at what happens when we try increasing these universal domains/ranges in a natural manner. Recall that a Banach lattice has a *Levi* norm if every norm bounded upward directed set of positive elements has a supremum.

Theorem 1.7. *The following conditions on a Banach lattice Y are equivalent:*

- (1) *Y has a Levi norm.*
- (2) *Whenever X is isomorphic to an AL-space, $\mathcal{L}(X, Y) = \mathcal{L}^r(X, Y)$.*
- (3) *Whenever X is isomorphic to an AL-space, $\mathcal{L}(X, Y)$ is a vector lattice.*

In [9], the fact that (1) $\Rightarrow$ (3) was established under the slightly stronger hypothesis that Y was a KB-space. Synnatzschke established the full force of that implication in [13]. Clearly (3) $\Rightarrow$ (2) and the fact that (2) $\Rightarrow$ (1) is Theorem 3.5 of [3].

At the other extreme we have:

Theorem 1.8. *The following conditions on a Banach lattice X are equivalent:*

- (1) *X is atomic with an order continuous norm.*
- (2) *If Y is isomorphic to an AM-space then $\mathcal{L}(X, Y) = \mathcal{L}^r(X, Y)$.*
- (3) *If Y is isomorphic to an AM-space then $\mathcal{L}(X, Y)$ is a lattice.*

2. THE ORDER STRUCTURE OF $\mathcal{L}^r(X, Y)$

The space of all regular operators does at least have a *generating cone*, i.e. every regular operator is the difference of two positive operators, but in general there is little more that can be said about its order structure. It certainly need not be a lattice.

Example 2.1. $\mathcal{L}^r(L_1([0, 1]), c)$ is not a lattice.

Proof. Consider the two operators $T, U : L_1([0, 1]) \rightarrow c$ defined by $Ux = \|x\|_1(1)$ and $Tx = \sum_{n=1}^{\infty} r_n(x)e_{2n}$. As $r_n(x) \rightarrow 0$ we actually have $Tx \in c_0$. It is clear that $U \geq T, 0$ so that T is regular. Suppose that S were the supremum in $\mathcal{L}^r(L_1([0, 1]), c)$ of T and 0 . For each $k \in \mathbb{N}$ let P_k denote the band projection of c onto the one-dimensional band generated by e_k . It is clear that if k is odd then $(I - P_k) \circ U \geq T, 0$. It follows that for odd k we have

$$(I - P_k) \circ U \chi_{[0,1]} \geq S \chi_{[0,1]} \geq 0.$$

It follows that the k 'th entry in $S \chi_{[0,1]}$ is zero. On the other hand, as $S \chi_{[0,1]} \geq S r_n = e_{2n}$ the even entries in $S \chi_{[0,1]}$ are at least 1. We therefore see that $S \chi_{[0,1]} \notin c$ and thus $\mathcal{L}^r(L_1([0, 1]), c)$ is not a lattice. $\square$

Definition 2.2. A Banach lattice has *property (*)* if, for every sequence (f_n) in X_+^* which converges $\sigma(X^*, X)$ to $f \in X_+^*$ as $n \rightarrow \infty$, we have $|f_n - f| \rightarrow 0$ for $\sigma(X^*, X)$ as $n \rightarrow \infty$.

The connection between this property and the order structure of spaces of regular operators is given by the following result of van Rooij, Theorem 8.2 of [12].

Theorem 2.3. A Banach lattice X has property (*) if and only if $\mathcal{L}^r(X, c)$ is a lattice.

Proof. Suppose that X has property (*) and that $T, U \in \mathcal{L}^r(X, c)$ with $U \geq T, 0$. For each $n \in \mathbb{N}$ define two sequences of functionals in X^* by $f_n(x) = (Tx)_n$ and $g_n(x) = (Ux)_n$ and define also another two functionals by $f(x) = \lim_{n \rightarrow \infty} (Tx)_n$ and $g(x) = \lim_{n \rightarrow \infty} (Ux)_n$, so that $g, g - f, g_n, g_n - f_n \geq 0$. In the weak* topology we have $\lim_{n \rightarrow \infty} (g_n - f_n) = g - f$, so property (*) tells us that $\lim_{n \rightarrow \infty} |g_n - f_n - g + f| = 0$. As we also have $\lim_{n \rightarrow \infty} g_n = g$ so that $\lim_{n \rightarrow \infty} |g_n - g| = 0$, we see that $\lim_{n \rightarrow \infty} |f_n - f| = 0$. As $||f_n| - |f|| \leq |f_n - f|$ it follows that $\lim_{n \rightarrow \infty} |f_n| = |f|$. We may therefore define $S : X \rightarrow c$ by $Sx = (|f_n|(x))$ and it is clear that $S = T^+$.

If $\mathcal{L}^r(X, c)$ is a lattice then take $f, f_1, f_2, \dots \in X^*$ with $f_n \rightarrow f$ for the weak* topology. Define $T : X \rightarrow c$ by

$$\begin{aligned}(Tx)_{2n} &= (f - f_n)(x) \\ (Tx)_{2n-1} &= 0.\end{aligned}$$

If $x \in X_+$ then $Tx \leq f(x)(1)$ so that $T \in \mathcal{L}^r(X, c)$ and therefore $|T|$ exists. By the same argument as used in Example 2.1 $(|T|x)_{2n} = |f - f_n|(x)$ whilst $(|T|x)_{2n-1} = 0$. Therefore $\lim_{n \rightarrow \infty} |f - f_n|(x) = 0$ for all $x \in X$ as required. $\square$

As in the first section, we can seek the range spaces (or domains) which always give us a vector lattice of regular operators or we can attempt to tackle the much more difficult problem of describing exactly which pairs of Banach lattices make $\mathcal{L}^r(X, Y)$ be a vector lattice. We look at the easier problems first. We have already seen that a Banach lattice Y is Dedekind complete if and only if, for every Banach lattice X , $\mathcal{L}^r(X, Y)$ is a vector lattice.

In fact in this case we can say more, namely that the lattice operations are given by the so-called *Riesz-Kantorovich* formulae. Dually, in Theorem 10.2 of [12], van Rooij established:

Theorem 2.4. *A Banach lattice X is atomic with an order continuous norm if and only if, for every Banach lattice Y , $\mathcal{L}^r(X, Y)$ is a vector lattice.*

Again the lattice operations are described by the Riesz-Kantorovich formulae.

There is still no complete description of when $\mathcal{L}^r(X, Y)$ is a vector lattice, but there has been some progress on this in recent years. The importance of condition (*) was established by van Rooij when he proved in Theorem 8.12 of [12] that:

Theorem 2.5. *If X and Y are Banach lattices and $\mathcal{L}^r(X, Y)$ is a vector lattice then either X has property (*) or Y is Dedekind σ -complete.*

Proof. We will suppose that X has property (*) and that $\mathcal{L}^r(X, Y)$ is a vector lattice and show that Y is Dedekind σ -complete. If Y were not Dedekind σ -complete there would be a principal ideal J in Y which was not Dedekind σ -complete and it can be shown that $\mathcal{L}^r(X, J)$ will be a vector lattice, so we may assume that $Y = C(K)$ for some compact Hausdorff space. In order to show that $C(K)$ is Dedekind σ -complete it suffices, by a theorem of Veksler and Gejler, to show that

any disjoint sequence (h_n) in $C(K)_+$ which is bounded above by χ_K has a supremum.

As X does not have property $(*)$ we can find $f_n, f \in X_+^*$ with $f_n \rightarrow f$ weak* yet such that there is $x_0 \in X_+$ with $|f - f_n|(x_0) \not\rightarrow 0$. By extracting a subsequence we can ensure that this last sequence is bounded away from 0 and hence that $|f - f_n|(x_0) \geq 1$ for all $n \in \mathbb{N}$. Now set $s_n = |f - f_n|(x_0)$ and set $g_n = s_n^{-1}f_n + (1 - s_n^{-1})f$. We still have $g_n \rightarrow f$ weak* and now $|g_n - f|(x_0) = 1$ for all $n \in \mathbb{N}$.

Define $T : X \rightarrow C(K)$ by

$$Tx = \sum_{n=1}^{\infty} (f(x) - g_n(x))h_n$$

noting that the sequence is uniformly convergent. As $Tx \leq f(x)\chi_K$ for all $x \geq 0$, T is regular and therefore has a modulus $|T|$. We claim that $|T|(x_0) = \sup\{f_n : n \in \mathbb{N}\}$.

Pick $m \in \mathbb{N}$ and let B_m denote the operator of pointwise multiplication by h_m on $C(K)$, then for each $x \in X$ we have $(B_m T)(x) = (f - f_m)(x) \times h_m^2$ so that $h_m |T|(x_0) = (B_m |T|)(x_0) = |B_m T|(x_0) = |f - g_m|(x_0) \times h_m^2 = h_m^2$. Thus $|T|(x_0)$ coincides with h_m on the set $\{k \in K : h_m(k) > 0\}$. Similarly, if $h \in C(K)_+$ and $hh_n = 0$ for all $n \in \mathbb{N}$ then $h \times |T|(x_0) = 0$. It follows that the continuous function $|T|(x_0)$ coincides on a dense subset of K with the pointwise supremum of the (h_n) so must be their supremum in $C(K)$. $\square$

A major stumbling block to further progress here is that we don't know exactly what property $(*)$ amounts to in a Banach lattice setting. We do have partial results. Theorem 3.1 of [4] actually establishes a result (apparently) slightly stronger than this:

Theorem 2.6. *If X is a Banach lattice which is atomic with an order continuous norm then it has property $(*)$. If X has property $(*)$ and (a) is Dedekind σ -complete or (b) is separable or (c) $X = C_0(\Sigma)$, where Σ is a locally compact Hausdorff space, then it is atomic with an order continuous norm.*

Putting together the preceding two results plus a careful analysis of the proof that $\mathcal{L}^r(X, Y)$ is a vector lattice when Y is Dedekind complete gives us:

Theorem 2.7. *If X and Y are Banach lattices and X is separable then $\mathcal{L}^r(X, Y)$ is a vector lattice if and only if either X is atomic with an order continuous norm or Y is Dedekind σ -complete.*

The separability allows us to go from property $(*)$, which is not obviously enough to guarantee that we have a lattice of operators, to

being atomic with an order continuous norm, which is. In order even to *state* a generalization of the last result, we need to introduce two new definitions...

Definition 2.8. The *density character* of a subset S of a topological space is the smallest cardinal $\mathfrak{a}$ such that S contains a dense subset of cardinality $\mathfrak{a}$.

Definition 2.9. A vector lattice Y is Dedekind $\mathfrak{a}$ -complete if every non-empty subset of cardinality at most $\mathfrak{a}$, which is bounded above, has a supremum, whilst we say that Y is Dedekind $<\mathfrak{a}$ -complete if every non-empty subset of cardinality strictly less than $\mathfrak{a}$, and which is bounded above, has a supremum.

Full details of the following theorem may be found in [6] whilst an outline of the proof may be found in [18].

Theorem 2.10. *Let X be a Banach lattice with an order continuous norm and Y be any Banach lattice. The following are equivalent:*

- (1) *Either X is atomic with an order continuous norm or Y is Dedekind $<\mathfrak{a}$ -complete where $\mathfrak{a}$ is the smallest cardinal that is greater than the density character of every order interval in X .*
- (2) *$\mathcal{L}^b(X, Y)$ is a Banach lattice under the regular norm in which all lattice operations satisfy the Riesz-Kantorovich formulae.*
- (3) *$\mathcal{L}^r(X, Y)$ is a vector lattice.*

The formulation of the preceding result involving $<\mathfrak{a}$ -completeness seems rather artificial. We could hope to replace it by $\mathfrak{b}$ -completeness where $\mathfrak{b}$ is the supremum of density characters of order intervals in X . There is no prospect of proving such a result unconditionally, but there is with a set-theoretic assumption.

Definition 2.11. A cardinal $\mathfrak{a}$ is *regular* if there is no cofinal subset $T \subset \mathfrak{a}$ with cardinality strictly less than $\mathfrak{a}$. An uncountable cardinal $\mathfrak{a}$ is termed *weakly inaccessible* if it is a regular limit cardinal. Their existence cannot be proved inside Zermelo-Fraenkel set theory together with the Axiom of Choice.

In [6], Elliott established:

Theorem 2.12. *Assume that there is no weakly inaccessible cardinal. Let X be a Banach lattice with an order continuous norm and Y be any Banach lattice. The following are equivalent:*

- (1) *Either X is atomic with an order continuous norm or Y is Dedekind $\mathfrak{a}$ -complete where $\mathfrak{a}$ is the smallest cardinal that is*

greater than or equal to the density character of every order interval in X .

- (2) $\mathcal{L}^b(X, Y)$ is a Banach lattice under the regular norm in which all lattice operations satisfy the Riesz-Kantorovich formulae.
- (3) $\mathcal{L}^r(X, Y)$ is a vector lattice.

He also showed that the equivalence of the three conditions in the theorem is actually equivalent to the non-existence of inaccessible cardinals.

Again, in these last cases, if $\mathcal{L}^r(X, Y)$ is a vector lattice then the lattice operations are all given by the Riesz-Kantorovich formulae. There is now enough evidence that I would lean towards the conjecture that if $\mathcal{L}^r(X, Y)$ is a vector lattice then the lattice operations in $\mathcal{L}^r(X, Y)$ are all given by the Riesz-Kantorovich formulae. I still have an open mind about whether or not it is true that if a single operator $T \in L^r(X, Y)$ has a modulus then it is given by the Riesz-Kantorovich formula.

What about weaker conditions on the order structure? The only obvious one that seems worth considering is the Riesz Separation Property.

Definition 2.13. We say that a Banach lattice X has the *countable interpolation property* if, given countable subsets A and C of X such that if $a \in A$ and $c \in C$ then $a \leq c$, then there is $b \in X$ such that $a \leq b \leq c$ for all $a \in A$ and $c \in C$.

The countable interpolation property is also known as the *Cantor property*. It is well-known that $C(K)$ has the Cantor property if and only if K is an F -space, i.e. the closures of two disjoint open F_σ sets are disjoint. This kind of property is quite strong but falls short of Dedekind σ -completeness. The following result was established by Dăneţ [5] and Wickstead [15].

Theorem 2.14. *The following conditions on a Banach lattice Y are equivalent:*

- (1) Y has the Cantor property.
- (2) For every separable Banach lattice X , $\mathcal{L}^r(X, Y)$ has the Riesz Separation Property.
- (3) $\mathcal{L}^r(c, Y)$ has the Riesz Separation Property.

3. WHEN ARE ALL ORDER BOUNDED OPERATORS REGULAR?

Remember that all regular operators are order bounded but that not all order bounded operators are regular. There will certainly be equality if Y is Dedekind complete, but we can do slightly better. There are no essentially new ideas needed to prove this result.

Theorem 3.1. *Let X and Y be Banach lattices such that whenever $\mathfrak{a}$ is the density character of an order interval in X then Y is Dedekind $\mathfrak{a}$ -complete, then $\mathcal{L}^r(X, Y) = \mathcal{L}^b(X, Y)$.*

In fact, in such a case $\mathcal{L}^b(X, Y)$ will actually be a lattice. There are very few general results in this area. The few that are known concern the special case when $Y = C(K)$, for K a compact Hausdorff space. The following result is Theorem 4.1 of [14]

Theorem 3.2. *If K is a compact Hausdorff space such that, for every Banach lattice X , $\mathcal{L}^r(X, C(K)) = \mathcal{L}^b(X, C(K))$ then there is $n \in \mathbb{N}$ such that, for any disjoint open sets $U_1, U_2, \dots, U_{n+1}$ in K , $\bigcap_{k=1}^{n+1} \overline{U_k} = \emptyset$.*

If we take two disjoint infinite Stonean spaces S_1 and S_2 and identify a non-isolated point in each of the spaces then we obtain a space K that is not Stonean (so that $C(K)$ is not Dedekind complete) yet has the property described in the theorem, for $n = 2$, so that every order bounded operator from any Banach lattice into $C(K)$ is actually regular (see [2].) By suitable choices of K we can ensure that $C(K)$ either is or is not Dedekind σ -complete. See [7] and [14] for details of these examples. In particular, let us note that it is possible for the order bounded and regular operators to coincide without being a vector lattice.

Just as there is a natural norm on the space of regular operators, under which it is norm complete, so also there is a natural norm on the space of order bounded operators, namely the *order bound norm* defined by

$$\|T\|_b = \inf\{M \in \mathbb{R} : \forall x \in X_+ \exists y \in Y_+ \text{ with } T([-x, x]) \subseteq [-y, y] \text{ and } \|y\| \leq M\|x\|\},$$

under which $\mathcal{L}^b(X, Y)$ is norm complete, see [16] for details.

If $\mathcal{L}^r(X, Y) = \mathcal{L}^b(X, Y)$ then the regular and order bound norms must be equivalent, although there are examples to show that they need not be equal. Their non-equivalence can sometimes be used to prove that not all order bounded operators are regular.

Example 3.3. For all $n \in \mathbb{N}$, $\mathcal{L}^r(c^n, c) = \mathcal{L}^b(c^n, c)$ but $\mathcal{L}^r(\ell_\infty(c^n), c) \neq \mathcal{L}^b(\ell_\infty(c^n), c)$.

The proof given in [16] uses the fact that we can find an operator T in $\mathcal{L}^r(c^n, c)$ with $\|T\|_b = 2$ and $\|T\|_r \geq n$. This operator T already

shows that the regular and order bound norms need not be equal even when $\mathcal{L}^r(X, Y) = \mathcal{L}^b(X, Y)$.

4. WHAT KIND OF DUALITY THEORY IS THERE FOR REGULAR OPERATORS?

If $T \in \mathcal{L}^r(X, Y)$ then we certainly have $T^* \in \mathcal{L}^r(Y^*, X^*)$ and $\mathcal{J} \circ T \in \mathcal{L}^r(X, Y^{**})$, where $\mathcal{J} : Y \rightarrow Y^{**}$ is the natural embedding of Y into its bidual.

Example 4.1. If $T : L_1([0, 1]) \rightarrow c_0$ is the bounded non-regular operator defined in Example 1.2 then both $T^* : c_0^* = \ell_1 \rightarrow L_1([0, 1])^* = \mathcal{L}_\infty([0, 1])$ and $\mathcal{J} \circ T : L_1([0, 1]) \rightarrow c_0^{**} = \ell_\infty$ are regular.

This already shows that some care needs to be taken with duality for regular operators. It turns out though that T^* and $\mathcal{J} \circ T$ both behave in pretty much the same way.

Theorem 4.2. *If X and Y are Banach lattices and $T \in \mathcal{L}(X, Y)$ then $T^* \in \mathcal{L}^r(Y^*, X^*)$ if and only if $\mathcal{J}_Y \circ T \in \mathcal{L}^r(X, Y^{**})$.*

Proof. Corollary 4.5 of [18]. □

If T is regular then both of the operators T^* and $\mathcal{J} \circ T$ are into a Dedekind complete range space so that both operators have a modulus. If T itself has a modulus, then it would be nice if we knew that $|T^*| = |T|^*$ or that $|\mathcal{J} \circ T| = \mathcal{J} \circ |T|$. Unfortunately neither of these equalities holds in general. The same holds, of course, for the positive and negative parts of these operators.

Example 4.3. Let K be an infinite compact Hausdorff space and let k_0 be any non-isolated point in K . Define $T : C(K) \rightarrow C(K)$ be defined by $Tf = f - f(k_0)\chi_K$. As k_0 is not isolated, for any $f \in C(K)_+$, we have $f = \sup\{g \in C(K) : 0 \leq g \leq f \text{ and } g(k_0) = 0\}$. Clearly $I \geq T, 0$, where I is the identity operator on $C(K)$. On the other hand if $U \geq T, 0$ and $f \geq 0$ then Uf is an upper bound for $\{Tg : 0 \leq g \leq f\}$ and hence for $\{Tg : 0 \leq g \leq f \text{ and } g(k_0) = 0\} = \{g : 0 \leq g \leq f \text{ and } g(k_0) = 0\}$ so that $Uf \geq f$. Thus $U \geq I$ and it follows that T^+ exists and is equal to I . If ε_{k_0} is the linear functional mapping f to $f(k_0)$ then $T^*(\varepsilon_{k_0})(f) = \varepsilon_{k_0}(Tf) = (Tf)(k_0) = 0$ for all $f \in C(K)$ so that $T^*(\varepsilon_{k_0}) = 0$. But $(T^*)^+(\varepsilon_{k_0}) = \sup\{T^*\mu : 0 \leq \mu \leq \varepsilon_{k_0}\} = 0$ as ε_{k_0} is extremal. As $(T^+)^*(\varepsilon_{k_0}) = I^*(\varepsilon_{k_0}) = \varepsilon_{k_0}$, we see that $(T^*)^+ \neq (T^+)^*$ and it follows that $|T^*| \neq |T|^*$.

We do, however, have:

Theorem 4.4. *Suppose that X and Y are Banach lattices and T is a regular operator from X into Y such that the modulus $|T|$ of T exists in $\mathcal{L}^r(X, Y)$. Then $\mathcal{J}_Y \circ |T| = |\mathcal{J}_Y \circ T|$ if and only if $|T|^* = |T^*|$.*

Proof. Theorem 4.7 of [18]. □

Again, it is of interest to know conditions on X and Y which guarantee that these equalities do hold.

Definition 4.5. The pair of Banach lattices (X, Y) has the *invariant modulus property* if, for every $T \in \mathcal{L}^r(X, Y)$ for which $|T|$ exists in $\mathcal{L}^r(X, Y)$, we have $|T|^* = |T^*|$ and, equivalently, $|\mathcal{J} \circ T| = \mathcal{J} \circ |T|$.

A complete characterization of the invariant modulus property again eludes us, but if we assume that Y is Dedekind σ -complete then we are very close to one. This is more useful than it might seem as without Dedekind σ -completeness there will be few operators in $\mathcal{L}^r(X, Y)$ with a modulus. The first, and most well known, result in this area is due to Synnatzschke, [13], who established that (1)(b) $\Rightarrow$ (2). The remainder of the proof may be found in Theorem 4.1 of [4]. An alternative short proof of Synnatzschke's result may be found in [18].

Theorem 4.6. *Let X and Y be Banach lattices with Y Dedekind σ -complete and consider the following three conditions:*

- (1) *At least one of following two conditions holds:*
 - (a) *X is atomic with an order continuous norm.*
 - (b) *The norm on Y is order continuous.*
- (2) *The pair (X, Y) has the invariant modulus property.*
- (3) *At least one of following two conditions holds:*
 - (a) *The lattice operations in X^* are weak* sequentially continuous.*
 - (b) *The norm on Y is order continuous.*

We always have (1) implies (2) implies (3) and if X is Dedekind σ -complete or is separable or $X = C_0(\Sigma)$, where Σ is a locally compact Hausdorff space, then all three conditions are equivalent.

If we allow the range or domain to vary our results are much more complete, see Corollary 4.2 and Theorem 4.3 of [4].

Corollary 4.7. *A Dedekind σ -complete Banach lattice Y has an order continuous norm if and only if for every Banach lattice X the pair (X, Y) has the invariant modulus property.*

Theorem 4.8. *A Banach lattice X is atomic with an order continuous norm if and only if for every Banach lattice Y the pair (X, Y) has the invariant modulus property.*

5. WHAT HAPPENS FOR THE CLASSICAL BANACH LATTICES?

The results above tell us what happens in quite a few cases, but there certainly are gaps in what the general theory tell us. If we limit our attention to some of the classical Banach lattices then we would hope to establish more. Even here our knowledge remains incomplete.

The most notable gaps in our knowledge concern operators from $C(K)$ into $C(\Omega)$ and into c_0 . The latter case is relatively straightforward, at least to state. Many of the results in the first part of this section may be found in [19]. This is Xiong's Theorem 2.8.

Theorem 5.1. *If K is an infinite compact Hausdorff space then $\mathcal{L}^r(C(K), c_0) \neq \mathcal{L}(C(K), c_0)$.*

The proof of this theorem differs greatly depending on whether or not K contains a non-trivial convergent sequence. Such sequences are clearly important in this context.

Definition 5.2. If K is a compact Hausdorff space then we define X_ℓ to be the set of points $k \in K$ for which there is a *sequence* of distinct points in K which converges to k .

Definition 5.3. If $n \in \mathbb{N}$ then a compact Hausdorff space K is termed an $X(n)$ *space* if every sequence in K has a convergent subsequence and K_ℓ has precisely n members.

Theorems 2.12 and 2.13 of [19] establish the next two results.

Theorem 5.4. *If K is a compact Hausdorff space then the following are equivalent:*

- (1) *For some $p \in \mathbb{N}$, K is an $X(p)$ -space.*
- (2) *For every compact Hausdorff space Ω , $\mathcal{L}(C(K), C(\Omega)) = \mathcal{L}^r(C(K), C(\Omega))$.*
- (3) *For some compact Hausdorff space Ω in which there is a non-trivial convergent sequence, $\mathcal{L}(C(K), C(\Omega)) = \mathcal{L}^r(C(K), C(\Omega))$.*
- (4) $\mathcal{L}(C(K), C([0, 1])) = \mathcal{L}^r(C(K), C([0, 1]))$.
- (5) $\mathcal{L}(C(K), c) = \mathcal{L}(C(K), c)$.

Theorem 5.5. *If K is a compact Hausdorff space then the following are equivalent:*

- (1) *K is an $X(1)$ -space.*
- (2) *For every compact Hausdorff space Ω , $\mathcal{L}(C(K), C(\Omega)) = \mathcal{L}^r(C(K), C(\Omega))$ and $\|T\| = \|T\|_r$ for all $T \in \mathcal{L}(C(K), C(\Omega))$.*
- (3) *For some compact Hausdorff space Ω in which there is a non-trivial convergent sequence, $\mathcal{L}(C(K), C(\Omega)) = \mathcal{L}^r(C(K), C(\Omega))$ and $\|T\| = \|T\|_r$ for all $T \in \mathcal{L}(C(K), C(\Omega))$.*

- (4) $\mathcal{L}(C(K), C([0, 1])) = \mathcal{L}^r(C(K), C([0, 1]))$ and $\|T\| = \|T\|_r$ for all $T \in \mathcal{L}(C(K), C([0, 1]))$.
- (5) $\mathcal{L}(C(K), c) = \mathcal{L}(C(K), c)$ and $\|T\| = \|T\|_r$ for all $T \in \mathcal{L}(C(K), c)$.

Finally, we address the question of when the space of regular operators can itself be one of the classical Banach lattices. The first result is contained in Theorem 2.1 of [17].

Theorem 5.6. *If X and Y are Banach lattices, neither of which is the zero space, then $\mathcal{L}^r(X, Y)$ is an AL-space under the regular norm if and only if X is an AM-space and Y is an AL-space.*

Things are a little trickier when we want $\mathcal{L}^r(X, Y)$ to be an AM-space.

Definition 5.7. If $\mathfrak{a}$ is an infinite cardinal then a Banach lattice X has an $\mathfrak{a}$ -Fatou norm if whenever $A \subset X_+$ is upward directed with supremum b and the cardinality of A is at most $\mathfrak{a}$, then $\|b\| = \sup\{\|a\| : a \in A\}$.

The following result is taken from [6], after a partial result was established in Theorem 2.2 of [17].

Theorem 5.8. *Let X and Y be non-zero Banach lattices, and let $\mathfrak{a}$ be the smallest cardinal that exceeds the density character of every order interval in X , then the following are equivalent:*

- (1) *The following three conditions are all satisfied:*
 - (a) *X is an AL-space.*
 - (b) *Y is an AM-space.*
 - (c) *Either X is atomic or Y is Dedekind $<\mathfrak{a}$ -complete and has the $<\mathfrak{a}$ -Fatou property.*
- (2) *$\mathcal{L}^r(X, Y)$ is an AM-space under the operator norm.*
- (3) *$\mathcal{L}^r(X, Y)$ is an AM-space under the regular norm.*

Proof. Theorem 5.8 of [18]. □

Again, it is worth recording that Elliott has given an example in [6] to show that the rather simpler version that one might hope for where “ $<\mathfrak{a}$ ” is replaced by “ $\mathfrak{a}$ ” is also equivalent to the non-existence of inaccessible cardinals. A simpler version of the preceding result may be easier to appreciate. The fact that AM-spaces with a Fatou norm are of the form $C_0(\Sigma)$ follows from a result of Nakano that may be found in [10] and [11].

Corollary 5.9. *If Y is a Banach lattice such that $\mathcal{L}^r(X, Y)$ is an AM-space for all AL-spaces X , then Y is isometrically order isomorphic to $C_0(\Sigma)$ for some locally compact extremally disconnected space Σ .*

By way of contrast, things are not at all as interesting in the L_p case.

Theorem 5.10. *If X and Y are non-trivial Banach lattices and $\mathcal{L}^r(X, Y)$ is isometrically isomorphic to an L_p -space, where $1 < p < \infty$, then either X or Y is one-dimensional.*

Proof. Theorem 5.11 of [18]. □

There are also isomorphic versions of the last three results.

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